UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra
5 September 2005: Morning Session, 9:00-12:00

(1) Consider the ring $R = \mathbb{Z}[X]$.
  (a) Is R a principal ideal domain? (Prove your answer.)
  (b) Find a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ has 4 elements.
  (c) Show that
$$M = \mathbb{Z}[X]/(X^2 - 5X - 2, 4X + 2).$$
  is a finitely generated abelian group. Determine its structure. (Here $(X^2 - 5X - 2, 4X + 2)$ is the ideal of $\mathbb{Z}[X]$ generated by the elements $X^2 - 5X - 2$ and $4X + 2$.)

(2) Suppose that V and W are $\mathbb{C}$-vector spaces of dimension n and m respectively.
  (a) Suppose that $e_1, e_2, \ldots, e_n$ is a basis of V and $e_1^\star, e_2^\star, \ldots, e_n^\star$ is the dual basis in $V^\star$. We define a map $\Theta : \mathrm{Hom}_{\mathbb{C}}(V^\star, W) \to V \otimes W$ by
$$\Theta(\phi) = \sum_{i=1}^{n} e_i \otimes \phi(e_i^\star).$$
  Prove that Θ is a linear isomorphism.
  (b) Prove that the map Θ does not depend on the choice of the basis $e_1, \ldots, e_n$ (as long as we choose $e_1^\star, \ldots, e_n^\star$ dual to it).
  (c) Suppose that $x \in V \otimes W$. Show that one can choose a basis $e_1, \ldots, e_n$ of V and a basis $f_1, \ldots, f_m$ of W such that
$$x = e_1 \otimes f_1 + e_2 \otimes f_2 + \cdots + e_r \otimes f_r$$
  for some r. (You could use for example part (a).)

(3) For which primes p and positive integers n is every p-Sylow subgroup of the symmetric group Sym_n commutative?

(4) Suppose that A is an invertible complex 3×3 matrix such that A and A^2 are conjugate. What are the possible Jordan normal forms of A.

(5) Let q be a prime power and m an integer. For the field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$ we have the trace map T and the norm map N defined by $T(x) := \sum_{i=0}^{m-1} x^{q^i}$ and $N(x) := \prod_{i=0}^{m-1} x^{q^i}$, respectively. These maps have domain $\mathbb{F}_{q^m}$ and codomain $\mathbb{F}_q$.
  (a) Prove that N is onto.
  (b) Give necessary and sufficient conditions on q and m for the restriction of N to $\mathbb{F}_q$ to be onto.
  (c) Prove that T is onto.
  (d) Give necessary and sufficient conditions on q and m for the restriction of T to $\mathbb{F}_q$ to be onto.

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5 September 2005: Afternoon Session, 2:00-5:00

(1) Suppose that $A, B, C \in M_n(\mathbb{C})$ are matrices such that A commutes with B and C, but B and C do not commute. Prove that the minimum polynomial of A has degree at most $n-1$.

(2) (a) Prove that if a group H acts on a set and K is a normal subgroup of H, then H leaves invariant the set of fixed points of K.
(b) Suppose that the finite group G acts transitively on the set Ω. Let P be a p-Sylow subgroup of G for a prime number p. Prove that $N(P)$, the normalizer of P, acts transitively on the set of fixed points of P on Ω (when this set of fixed points is nonempty).

(3) Let p be a prime and let $\mathbb{F}_q$ be a finite field of $q = p^m$ elements.
(a) Prove that if $x \in \mathrm{GL}_2(\mathbb{F}_q)$ is a nonidentity element of order a power of p, then its order is p and its minimal polynomial is $(t-1)^2$. Prove that such an element fixes a unique 1-dimensional space.
(b) Prove that Ω, the set of 1-dimensional subspaces in $\mathbb{F}_q^2$, has cardinality $1+q$.
(c) Prove that an element of $\mathrm{GL}_2(\mathbb{F}_q)$ fixes exactly one point of Ω if and only if it has order divisible by p.

(4) Let E_1, E_2 be subfields of the algebraic closure $\bar{F}$ of the field F. Assume that E_i/F is a finite degree Galois extension for $i = 1, 2$ and that $\mathrm{Gal}(E_1/F) \cong \mathrm{Dih}_{10}$ and $\mathrm{Gal}(E_2/F) \cong \mathrm{Dih}_{14}$. (Here Dih_{2n} is the dihedral group of order $2n$.) Let E_1E_2 be the subfield of $\bar{F}$ generated by E_1 and E_2.

There is a homomorphism of groups

$$\phi = (r_1, r_2) : \mathrm{Gal}(E_1E_2/F) \to \mathrm{Gal}(E_1) \times \mathrm{Gal}(E_2),$$

where r_i is the restriction to the subfield E_i/F, for $i = 1, 2$.
(a) Prove that ϕ is a monomorphism.
(b) Prove that $|\mathrm{Im}(\phi)| = 2 \cdot 5 \cdot 7$ or $2^2 \cdot 5 \cdot 7$.
(c) In case $|\mathrm{Im}(\phi)| = 2 \cdot 5 \cdot 7$, prove that $\mathrm{Im}(\phi)$ is dihedral of order 70.

(5) Consider the $\mathbb{C}[X]$-modules $M_1 := \mathbb{C}[X]/(X^6 - X^2)$ and $M_2 := \mathbb{C}[X]/(X^9 - X^3)$. Let $M = M_1 \otimes_{\mathbb{C}[X]} M_2$ be the tensor product of M_1 and M_2 as $\mathbb{C}[X]$-modules. Write M as a finite direct sum of cyclic modules of the form $\mathbb{C}[X]/((X-a)^m)$, $a \in \mathbb{C}$, $m \in \mathbb{N}$.