UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology
3 January 2005: Morning Session, 9:00-12:00

Problem 1 Assume that $f : X \to Y$ is a smooth map between two smooth manifolds.

(a) Must there exist a smooth manifold Z, an immersion $g : X \to Z$, and a submersion $h : Z \to Y$ such that $f = h \circ g$? Why or why not?

(b) Must there exist a smooth manifold Z, a submersion $g : X \to Z$, and an immersion $h : Z \to Y$ such that $f = h \circ g$? Why or why not?

Problem 2 In this problem S^n means $\{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$, and $f : S^n \to S^n$ is a continuous map that satisfies $f(-x) = f(x)$ for every $x \in S^n$. For which n can one infer from these assumptions that there is a $y \in S^n$ such that $f(y) = y$?

Problem 3 Let X be a Hausdorff space and let $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ be a decreasing sequence of compact, connected subsets of X. Prove that $\bigcap_{k=1}^{\infty} C_k$ is also compact and connected.

Problem 4 Let X be the union of the unit sphere in $\mathbb{R}^3$ and the coordinate axes x, y, z. Calculate the integral homology of $\mathbb{R}^3 - X$.

Problem 5 (a) Prove that a connected, metrizable space with at least two points must be uncountable.

(b) Prove or disprove the same result with “metrizable” changed to “compact, Hausdorff.”

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3 January 2005: Afternoon Session, 2:00-5:00

Problem 1 Let X be a compact oriented 2-dimensional manifold X without boundary, and let Y be obtained by removing a small open disk from X. For which such X's is there a continuous retraction of Y onto its boundary circle?

Problem 2 Consider the ellipsoid $X \subseteq \mathbb{R}^3$ and the sphere $Y \subseteq \mathbb{R}^3$ given by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = r^2,$$

with $a > b > c > 0$ fixed. For what values of r is $X \cap Y$ a smooth manifold? What is its dimension?

Problem 3 Consider the space X obtained from $S^1 \times [0, 1]$ by identifying $(x, 0) \sim (x^3, 1)$ for $x \in S^1 = \{z \in \mathbb{C} | \ |z| = 1\}$. Calculate $\pi_1(X)$, and show how you arrived at your answer.

Problem 4 Prove that, for any locally compact, Hausdorff space, the following two statements are equivalent
(a) X is compact.
(b) Whenever X is homeomorphic to a subspace Y of a Hausdorff space Z, then Y is closed in Z.

Problem 5 Take the space X obtained from the oriented compact 2-manifold (without boundary) of genus 2 by removing n points. How many isomorphism classes of 2-fold (connected) covers of X are there?