UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology
5 May 2005: Morning Session, 9:00-12:00

Problem 1 Let $f : X \to Y$, $g : Y \to Z$ be regular coverings. Is it true that the composition $gf : X \to Z$ is necessarily a regular covering? Prove or give a counterexample.

Problem 2 A *two-point compactification* of a Hausdorff space X is a compact Hausdorff space Y such that X is a dense subspace of Y and $Y - X$ consists of exactly two points. Prove that no two-point compactification of the Euclidean plane $\mathbb{R}^2$ exists.

Problem 3 Show for which $n = 1, 2, 3, \ldots$ the following statement holds: If a homeomorphism $f : S^n \to S^n$ preserves orientation, then f has a fixed point.

Problem 4 The *polar coordinate map* from $\mathbb{R}^2$ to itself is defined by

$$f(u, v) = (u \cos v, u \sin v).$$

Find and prove a necessary and sufficient condition (as simple as possible) for a curve C in the plane (i.e., a 1-dimensional smooth submanifold of $\mathbb{R}^2$) to have $f^{-1}(C)$ also a smooth manifold.

Problem 5 Consider the space X obtained from the triangle in $\mathbb{R}^3$ given by the conditions $x + y + z = 1$, $x, y, z \geq 0$ by identifying

$$(x, 1 - x, 0) \sim (x, 0, 1 - x),$$

$$(0, y, 1 - y) \sim (1, 0, 0)$$

for all $x, y \in [0, 1]$. Calculate the homology of X.

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5 May 2005: Afternoon Session, 2:00-5:00

Problem 1 Suppose $f : X \to Y$ is a smooth immersion between smooth manifolds of the same dimension. Given that X is compact and Y is connected, prove that Y is also compact.

Problem 2 Let $S^1 = \{z \in \mathbb{C} | \ ||z|| = 1\}$. Consider the space Z obtained from $S^1 \times [0,1]$ by identifying $(z,1) \sim (iz,1)$ and $(t,0) \sim (e^{\pi i/3}t, 0)$ for $z, t \in S^1$. Calculate $\pi_1(Z)$.

Problem 3 Prove or refute: If p is a point in a compact Hausdorff space X then there exists a continuous real-valued function $f : X \to \mathbb{R}$ that vanishes at p and nowhere else.

Problem 4 Construct an example of a 5-sheeted cover of the figure 8 which is not regular.

Problem 5 A continuous map $f : X \to Y$ between topological spaces is *closed* if, for every closed set C in X, the image $f(C)$ is closed in Y. For each of the following three sorts of maps, either prove that all such maps are closed or give a counterexample.

1. A linear map from one Euclidean space to another.
2. A covering projection.
3. A product projection $X \times Y \to X$ where Y is compact.