UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology
10 September 2005: Morning Session, 9:00-12:00

Problem 1 A *compactification* of a topological space X is a compact topological space Y with a homeomorphism from X to a dense subspace of Y. Prove that, when X is metrizable, the following two statements are equivalent.

1. X has a metrizable compactification.

2. X has a countable dense subset.

[Hint: The product of countably many copies of $[0, 1]$, with the product topology is metrizable. You may use this fact without proving it.]

Problem 2 Define a topological space X by taking a disjoint union of two copies of the torus $S^1 \times S^1$, choosing one point in each copy, and identifying those points (the topology on X is the quotient topology). Is this space homotopy equivalent to a 2-manifold? Prove or disprove.

Problem 3 (a) Suppose X is a Hausdorff space, equipped with a continuous binary operation $f : X \times X \to X$ that has an identity element e. Prove that the connected component of X that contains e is closed under the operation f.

(b) Does the result in (a) remain true if, instead of assuming that f is continuous, we assume only that, for each fixed $a \in X$, the function $x \mapsto f(a, x)$ is continuous? Prove it or give a counterexample.

Problem 4 Find the homology of the space which is a union of a (2-dimensional) sphere S in $\mathbb{R}^3$ and the surface of a cube whose corners are on S.

Problem 5 Let $f : M \to \mathbb{R}$ be a smooth, real-valued function on a smooth manifold M, and let $p \in M$ be a critical point of f. It is called a *regular* critical point if the matrix of second derivatives of f at p, with respect to a local coordinate system, is nonsingular.

(a) Prove that this notion of regularity is independent of the choice of the local coordinate system.

(b) Give an explicit example of a smooth function from $S^1 \times S^1 \to \mathbb{R}$ such that all of its critical points are regular.

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Problem 6 Let $S^1 = \{z \in \mathbb{C} | \ ||z|| = 1\}$. Consider the space $X = S^1 \times S^1$ (with the product topology) and the map $f : S^1 \to X$ given by $z \mapsto (z^2, z^4)$, $z \in S^1$. Does there exist a map $g : X \to S^1$ such that the composition $f \circ g$ has no fixed points?

Problem 7 Assume that X and Y are smooth manifolds of dimensions m and n, respectively, and that f, g, h are smooth functions $X \to Y$. Let $A = \{x \in X : f(x) = g(x) = h(x)\}$. Give conditions on the differentials df, dg, and dh (linear maps of tangent spaces $T_a X \to T_{f(a)} Y$) at a point $a \in A$ that would ensure that A is locally (near a) a smooth submanifold of X? Under these conditions, what is the dimension of A (near a)?

Problem 8 Consider, on $Z = S^1 \times S^1 \times \{0, 1\}$, the smallest equivalence relation $\sim$ such that $(z, 1, 0) \sim (z, 1, 1)$ for all $z \in S^1$. Let $X = Z/\sim$ (with the quotient topology).

(a) Prove that there exists a space Y, a homotopy equivalence $f : Y \to S^1$, and a covering $p : Y \to X$ which is homotopic to the map $y \mapsto (f(y), 1, 0)$.

(b) Is the covering p regular?

Problem 9 Assume that X is a locally compact Hausdorff space with infinitely many points. Prove that it has an infinite, strongly discrete subspace D. ("Strongly discrete" means that one can choose, for each point in D, a neighborhood of it in X, such that the chosen neighborhoods of any two distinct points in D are disjoint.)

Problem 10 Let $f, g : X \to Y$ be two continuous maps. Consider the space Z which is obtained from $Y \amalg (X \times [0, 1])$ by identifying $(x, 0) \sim f(x)$, $(x, 1) \sim g(x)$. Prove that there is a long exact sequence of the following form:

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \xrightarrow{\subset_*} H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

[Hint: use the long exact sequence associated with the inclusion $Y \subset Z$.]