UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra
3 January 2006: Morning Session, 9:00-12:00

(#1). This problem deals with the following three matrices of integers:

$$A := \begin{pmatrix} 2 & 8 \\ 2 & -6 \\ 4 & -42 \end{pmatrix} \quad , \quad B := \begin{pmatrix} 2 & 8 \\ 2 & -6 \end{pmatrix} \quad , \quad C := \begin{pmatrix} 2 & -6 \\ 4 & -42 \end{pmatrix}$$

Note that the rows of A come from those of B and C.

Let $F = \mathbf{Z}^2$. Given a $k \times 2$ matrix M of integers, we let $R(M)$ denote the $\mathbf{Z}$-span of its rows. Thus $R(M) \subseteq F$. For each of the following quotient groups, determine its isomorphism type as a direct sum of cyclic groups of explict orders.

(a). $F/R(A)$,

(b). $F/R(B)$,

(c). $F/R(C)$,

(d). $R(A)/R(C)$,

(e). $R(A)/R(B)$.

(#2). Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of degree 6. Let $K = \mathbf{Q}(\sqrt{2})$.

(a). Show that, considered as a polynomial in $K[x]$, $f(x)$ either remains irreducible or splits as a product of two polynomials of degree 3.

(b). Prove that if $f(x)$ splits in $K[x]$, then in fact it splits as the product of two *irreducible* cubics in $K[x]$.

(#3). Given an integer $n \geq 3$, prove that the $n-2$ permutations

$$(1,2,3), \quad (2,3,4), \quad (3,4,5), \quad \ldots \quad , \quad (n-2, n-1, n)$$

generate Alt_n, the alternating group of degree n.

(#4). Let V be the vector space of all real polynomials of degree ≤ 2, so that $\dim_{\mathbf{R}} V = 3$. Pick r distinct real numbers

$$a_1 \ < \ a_2 \ < \ \ldots \ < \ a_{r-1} \ < \ a_r,$$

and consider the symmetric bilinear form on V defined by:

$$B : V \times V \longrightarrow \mathbf{R} \quad , \quad B\big(f, g\big) \ = \ \sum_{i=1}^{r} f(a_i) \cdot g(a_i).$$

Note: If you want to make explicit calculations in what follows, take for example $a_1 = 1$, $a_2 = 2$, ... However ideally your arguments shouldn't depend on the particular choice of the a_i.

(a). Show that B is positive-semidefinite.

(b). Prove that if $r \leq 2$, then B is degenerate. What is its kernel? [Recall that the *radical* or *kernel* of B is by definition $\ker(B) = \{f \in V \mid B(f, g) = 0 \ \forall g \in V\}$.]

(c). Prove that if $r \geq 3$, then B is positive definite.

(d). State and prove a generalization of (a), (b) and (c) for the quadratic form defined by the same formula when V is replaced by the $(d+1)$-dimensional vector space of real polynomials of degree $\leq d$.

(# 5). This problem deals with polynomials in $\mathbf{F}_2[x]$, $\mathbf{F}_2$ being the field with two elements. We set

$$d(x) \ := \ x^8 + x + 1 \ \in \mathbf{F}_2[x].$$

(a). Prove that $d(x)$ is the g.c.d. of the polynomials

$$f(x) = x^{10} + x^9 + x^8 + x^3 + 1$$

and

$$g(x) = x^{11} + x^9 + x^8 + x^4 + x^3 + x^2 + 1$$

in $\mathbf{F}_2[x]$.

(b). Prove that $d(x)$ is not divisible by the square of an irreducible polynomial in $\mathbf{F}_2[x]$.

(c). Prove that $\mathbf{F}_2[x]/(d(x))$ is isomorphic (as a ring) to a direct sum of fields.

UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra

3 January 2006: Afternoon Session, 2:00-5:00

(#1). Let G be a finite solvable group. Suppose that $x, y, z \in G$ have pairwise relatively prime orders and that $xyz = 1$. Prove that

$$x = y = z = 1.$$

HINT: Consider first the case G abelian.

(#2). Let R be a commutative ring with 1, and let $I, J \subseteq R$ be ideals. Recall that $IJ \subseteq R$ is the ideal generated by all products uv for $u \in I$ and $v \in J$, and that $I+J \subseteq R$ is the ideal consisting of all sums $u + v$ with $u \in I$ and $v \in J$.

(a). Prove that if $I + J = R$, then $IJ = I \cap J$.

(b). Give an example of a ring R, and ideals $I, J \subseteq R$ such that $IJ \neq I \cap J$.

(c). Assuming again that $I + J = R$, show that given any $a, b \in R$, there exists an element $x \in R$ such that

$$x \equiv a \pmod{I} \quad \text{and} \quad x \equiv b \pmod{J}.$$

(Recall that $x \equiv a \pmod{I}$ means by definition that $x - a \in I$.)

(#3). A *flag* $V_\bullet$ in $\mathbf{R}^n$ is a chain of vector subspaces

$$(0) \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{n-1} \subseteq V_n = \mathbf{R}^n$$

with $\dim V_i = i$. We denote by $\mathbf{F}$ the set of all such flags.

(a). Let $G = GL(n, \mathbf{R})$ be the group of $n \times n$ invertible matrices with real entries. Show that the natural left action of G on $\mathbf{R}^n$ determines a transitive action of G on $\mathbf{F}$.

(b). Let $B \subseteq G$ be the subgroup of all upper-triangular matrices (with arbitrary non-zero entries on the diagonal). Show that there is a one-to-one correspondence between the set of all cosets gB of B in G and the set $\mathbf{F}$ of all flags, in such a way that the action of G on cosets corresponds to the action of G on $\mathbf{F}$.

(#4). Let V be a finite dimensional vector space over a field F and let $T \in \mathrm{End}_F(V)$ be an endomorphism of V. Suppose that T has minimal polynomial $p(x)$, where p is a monic irreducible polynomial in $F[x]$.

(a). Let $L \subseteq \mathrm{End}_F(V)$ be the subalgebra generated by F and T, i.e. the set of all polynomial expressions $\sum_i c_i T^i, c_i \in F$. Prove that L is a field, and that V is an L-vector space.

(b). Let

$$C(T) = \{S \in \mathrm{End}_F(V) \mid ST = TS\}.$$

Prove that the subring $C(T)$ is isomorphic as a ring to the matrix algebra $\mathrm{Mat}_{r\times r}(L)$ for some integer r.

(# 5). Let $L \supset k$ be a finite Galois extension of fields, with Galois group $G = \mathrm{Aut}_k(L)$. Let

$$k \subseteq E \subseteq L$$

be an intermediate field, corresponding to the subgroup $H \subseteq G$ under the Galois correspondence. Prove that

$$\mathrm{Aut}_k(E) \cong N_G(H)/H,$$

where $N_G(H) = \{\sigma \in G \mid \sigma H \sigma^{-1} = H\}$ is the normalizer of H in G.