UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra

6 May 2006: Morning Session, 9:00-12:00

(AM1). Recall that any two finite fields of the same cardinality are isomorphic. With this in mind, let $\mathbf{F}_2$ be the field with two elements. Exhibit explicitly an isomorphism between

$$\frac{\mathbf{F}_2[x]}{(x^3+x+1)} \quad \text{and} \quad \frac{\mathbf{F}_2[y]}{(y^3+y^2+1)},$$

which are both fields having eight elements.

(AM2). This problem concerns an $n \times n$ matrix A over a field k that satisfies the equation

$$(*) \qquad A^3 \ = \ \mathrm{Id}.$$

(a). Prove that if $k = \mathbf{Q}$, then A may not be diagonalizable, but if $k = \mathbf{Q}(\sqrt{-3})$ then A is diagonalizable.

(b). Now suppose that $k = \mathbf{F}_3$ is the field with three elements. List (with a brief explanation) all conjugacy classes over $\mathbf{F}_3$ of 5×5 matrices A satisfying (*). [Hint: show first that A can be put into Jordan canonical form over $\mathbf{F}_3$.]

(AM3). Suppose that V is a finite dimensional vector space over the field K and that f is an alternating bilinear form $V \times V \to K$. (Recall that f alternating means $f(x,x) = 0$ for all $x \in V$.)

(a). Prove that V is a direct sum

$$V \ \cong \ V_0 \perp H_1 \perp \cdots \perp H_r,$$

where

$$V_0 \ := \ \{x \in V | f(x,V) = 0\}$$

and the H_i are hyperbolic planes, i.e., 2-dimensional spaces with Gram matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (The notation $A \perp B$ means direct sum of subspaces such that $f(x,y) = 0$ for $x \in A, y \in B$.) We call $dim(V_0)$ the *nullity* of the form and $2r$ its *rank*.

(b). There is a natural $\mathbf{R}$-valued alternating bilinear form on $\mathbf{C}$, viz. $f(z,w) = \mathrm{Im}(z\bar{w})$ (Im means the imaginary part and bar means usual complex conjugation). A similar idea applies to $\mathbf{C}^n$, where we define

$$f\big((z_i),(w_i)\big) \ := \ \sum_i \mathrm{Im}(z_i\bar{w}_i).$$

Now consider the subspace $W \subseteq \mathbf{C}^3$ consisting of all vectors (z_1, z_2, z_3) such that $z_1 + z_2 + z_3 = 0$. Determine the nullity and rank of $f|_{W \times W}$.

(AM4). Let G be a finite group of order $2^5{\cdot}5{\cdot}31$. Prove that one of the following statements is true:

(a). A Sylow 31-group is normal; or

(b). There are 32 Sylow 31-groups; and if $P \neq Q$ are such, $|N_G(P) \cap N_G(Q)| = 5$.

(AM5). Let V and W be finite-dimensional vector spaces over a field k. Prove that there is a canonical isomorphism

$$\wedge^2\big(V \oplus W\big) \;\cong\; \wedge^2(V) \,\oplus\, (V \otimes W) \,\oplus\, \wedge^2(W).$$

("Canonical" means that the isomorphism should be independent of any choice of bases. The symbol $\wedge^r U$ means the r-th exterior tensor power of a vector space U.)

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6 May 2006: Afternoon Session, 2:00-5:00

(PM1) (a). Every invertible 2×2 matrix over the field F is a product of elementary matrices, i.e., matrices of the form

$$E_1 := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad E_2 := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \qquad E_3 := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_4 := \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \qquad E_5 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for some $t, u, a, b \in F$ with $a \neq 0, b \neq 0$. This fact comes from the theory of row operations and reduction; you don't need to prove it. Now suppose that F is a finite field of $q = p^m$ elements, for a prime number p and integer $m \geq 1$. Show that the order of each of the above matrices is a divisor of p or of $q - 1$.

(b). Let $GL(2, 5)$ be the group of invertible 2×2 matrices over the field of 5 elements. Suppose that $\psi : GL(2, 5) \to \mathbf{Z}_{21}$ is a homomorphism of groups. Prove that ψ is trivial, i.e. $Ker(\psi) = GL(2, 5)$.

(PM2) Let K be a field, $t \in K, t \neq 0$, and A the linear transformation from $V = K^2$ to itself with matrix

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

There is a linear tranformation, $A \otimes A : V \otimes V \to V \otimes V$, defined by

$$(A \otimes A)(u \otimes v) \ = Au \otimes Av$$

for $u, v \in V$. Determine the Jordan canonical form of $A \otimes A$. (It may depend on the field K.)

(PM3). Let $r = \sqrt{2} \in \mathbf{R}$ and let $s = \sqrt{3} \in \mathbf{R}$. Given rational numbers $a, b, c, d \in \mathbf{Q}$, not all zero, consider the real number

$$u = u_{a,b,c,d} = a + br + cs + drs \in \mathbf{R}.$$

The degree of u over $\mathbf{Q}$ depends on a, b, c and d. Find all the possibile values of this degree, and the precise conditions on a, b, c, d under which each of these possibilities occurs. Prove your answer.

(PM4). Let E be the splitting field of the polynomial $x^4 - 5$ over $\mathbf{Q}$. Show that E has degree 8 over $\mathbf{Q}$, and has Galois group isomorphic to the dihedral group of order 8. Prove that there exists a subfield of E which is not normal over $\mathbf{Q}$.

(PM5). Given an integer $d \in \mathbf{Z}$ which is not a square, denote by $\mathbf{Z}[\sqrt{d}]$ the ring

$$\mathbf{Z}[\sqrt{d}] \; = \; \left\{ a + b\sqrt{d} \;\middle|\; a, b \in \mathbf{Z} \right\}.$$

(a) Prove that if $d < 0$, then the group of units (the set of invertible elements) in $\mathbf{Z}[\sqrt{d}]$ is finite.

(b). Show that the group of units in $\mathbf{Z}[\sqrt{2}\,]$ is infinite.

Hint: Consider the norm $N(a + b\sqrt{d}) = a^2 - db^2$.