# UNIVERSITY OF MICHIGAN
# DEPARTMENT OF MATHEMATICS
## Qualifying Review Examination in Algebra

*4 September 2006: Morning Session, 9:00-12:00*

(#1). Let $V$ be an $n$-dimensional vector space over the complex numbers $\mathbf{C}$.

(a). Define what is meant by a Hermitian form $H$ on $V$.

Given a Hermitian form $H$ on $V$, set $g = \mathrm{Re}(H)$ and $\omega = \mathrm{Im}(H)$, so that

$$H(v,w) \;=\; g(v,w) \;+\; i \cdot \omega(v,w)$$

for all $v, w \in V$, where $g$ and $\omega$ are real-valued.

(b). Show that $g$ is a symmetric $\mathbf{R}$-bilinear form, and that $\omega$ is an alternating $\mathbf{R}$-bilinear form on $V$, where now $V$ is considered as a vector space of dimension $2n$ over $\mathbf{R}$. Prove moreover that

$$g(iv, iw) = g(v,w) \quad , \quad \omega(iv, iw) = \omega(v,w)$$

for all $v, w \in V$.

(c). Prove conversely that if $g$ is a symmetric $\mathbf{R}$-bilinear form on $V$ satisfying $g(iv, iw) = g(v,w)$ for all $v, w \in V$, then there exists a Hermitian form $H$ on $V$ such that $g = \mathrm{Re}(H)$.

(#2). Suppose $G$ is a finite group such that its automorphism group $\mathrm{Aut}(G)$ is solvable. Prove that $G$ itself is solvable.

(#3). (a). Prove that there are no $3 \times 3$ matrices $A$ with entries in $\mathbf{Q}$ such that

$$A^8 = I \quad \text{but} \quad A^4 \neq I.$$

(b). What happens if we look instead for $3 \times 3$ matrices with entries in $\mathbf{R}$ satisfying the same conditions?

(#4). (a). Suppose that $K_1, K_2$ are subfields of a field $L$ such that $L/(K_1 \cap K_2)$ is a finite field extension. Suppose that $L/K_1$ and $L/K_2$ are Galois extensions. Prove that $L/(K_1 \cap K_2)$ is a finite Galois extension.

(b). Let $x$ be a transcendental element over $\mathbf{Q}$. Suppose that $M$ is a subfield of $\mathbf{Q}(x)$ containing $\mathbf{Q}$, but not equal to $\mathbf{Q}$. Prove that $\mathbf{Q}(x)/M$ is a finite field extension.

(c). Prove that $L/K_1 \cap K_2$ is not a finite field extension, where $L = \mathbf{Q}(x)$, $K_1 = \mathbf{Q}(x^2)$ and $K_2 = \mathbf{Q}((x-1)^2)$. (Hence $K_1 \cap K_2 = \mathbf{Q}$ by (b)).

(#5). Suppose that $G$ a group of order $4n$ with $n$ odd, containing an element of order 4.

(a). Prove that the elements of order 4 in $G$ fall into at most 2 conjugacy classes.

(b). Prove that the elements of order 4 cannot form a single conjugacy class.

# UNIVERSITY OF MICHIGAN
# DEPARTMENT OF MATHEMATICS
## Qualifying Review Examination in Algebra

*4 September 2006: Afternoon Session, 2:00-5:00*

(#1). Let $R$ be a commutative ring with 1, and let $I \subseteq R$ be an ideal.

(a). The *radical* $\sqrt{I}$ of $I$ is defined to be the set

$$\sqrt{I} \;=\; \{a \in R \mid a^n \in I \text{ for some } n > 0 \text{ depending on } a\ \}.$$

Prove that $\sqrt{I}$ is an ideal, and that $R/\sqrt{I}$ has no non-zero nilpotents.

(b). Let $R = \mathbf{Z}$ and fix an integer $m \geq 2$. What is the radical $\sqrt{(m)}$ of the the principal ideal generated by $m$?

(c). Let $R = \mathbf{Q}[x, y]$ be the ring of polynomials in two variables with rational coefficients, and let $I = (x^2, y^5)$ be the ideal generated by $x^2$ and $y^5$. Find $\sqrt{I}$.

(#2). (a). Suppose that $G$ is a finite group acting on a finite set $X$. Prove that

$$\sum_{x \in X} |G_x|$$

is divisible by the group order $|G|$, where $G_x$ is the stabilizer group of $x \in X$.

(b). For a finite group $G$, prove that the number of pairs $(a, b)$ with $a, b \in G$ and $ab = ba$ is divisible by $|G|$.

(#3). Suppose that $A$ is a $3 \times 3$ matrix with complex entries such that $A$ is conjugate to $-A$. What are the possible Jordan normal forms of $A$.

(#4). Let $n$ be a positive integer.

(a). Prove that $\cos(\frac{2\pi}{n})$ is an algebraic number, and compute its degree over $\mathbf{Q}$ in terms of the Euler $\phi$-function.

(b). Find with proof all values of $n$ for which $\cos(\frac{2\pi}{n})$ is rational.

(#5). Let $V$ be vector space over a field $k$ of characteristic $\neq 2$, with $n = \dim V \geq 3$.

(a). Prove that there is a non-zero canonical map

$$V \otimes \Lambda^2 V \longrightarrow \Lambda^3 V.$$

("Canonical" means that the map you construct should be independent of any choice of bases.)

(b). Show that the map in (a) is surjective, but not an isomorphism.