UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Topology

7 January 2006: Morning Session, 9:00-12:00

1.a Show from scratch that if X is compact and locally connected then X has only finitely many (connected) components.
b. Show that the statement in (a) is false if X is not assumed to be locally connected.

2. The set of $n \times n$ matrices with real entries is denoted $M(n,n)$. We topologize $M(n,n)$ in the usual way by identifying it with $\mathbf{R}^{n^2}$ as follows : if $A = (a_{ij}) \in M(n,n)$ then let A correspond to the n^2-tuple $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{nn})$. Let $S = \{A \in M(n,n) : det(A) = 0\}$, where det is the determinant map.
a. Explain why $det : M(n,n) \to \mathbf{R}$ is a smooth function. (A formal proof is not necessary.)
b. Show that S is a closed subspace of $M(n,n)$ and that S has no interior points in $M(n,n)$.
c. If $n = 2$, show that every $r \in \mathbf{R}$ is a regular value for the restricted mapping $det : M(2,2) - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \to \mathbf{R}$.

3. Construct a path-connected space X for which $H_0(X;\mathbf{Z}) = \mathbf{Z}$, $H_1(X;\mathbf{Z}) = \mathbf{Z}/5\mathbf{Z}$ and $H_2(X;\mathbf{Z}) = \mathbf{Z}$. Explain why your construction works.

4. In the plane $\mathbf{R}^2$, identify the x-axis to a point and take the identification (i.e. quotient) topology. Answer the following questions and give your reasons.
a. Is the quotient space regular? (Recall that a space X is regular if for any $x \in X$ and any closed set $C \subset X$ not containing x, there are disjoint open sets U_1 and U_2 containing x and C, respectively.)
b. Does each point have a countable neighborhood basis (i.e. does the quotient space satisfy the First Axiom of Countability)?
c. Does the projection $(x, y) \to y$ induce a continuous map from the quotient space to $\mathbf{R}$?

5. Let K be a finite simplicial complex (or, if you prefer, a finite CW complex).
a. Define what is meant by the Euler characteristic $e(K)$ of K.
b. Let $\widetilde{K}$ be a finite simplicial (or cellular) covering of K. Show that $e(\widetilde{K}) = ne(K)$, where n is the order of the covering.
c. Use b to determine which orientable closed 2-manifolds can cover a torus. (A manifold is closed if is compact and has empty boundary.) You can assume what you know about the classification of orientable closed 2-manifolds.

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7 January 2006: Afternoon Session, 2:00-5:00

1. Consider $X = \mathbf{R}P^3 \cup I \cup \mathbf{R}P^3$, where $\mathbf{R}P^3$ is real projective 3-space, I is the unit interval and the second $\mathbf{R}P^3$ is another copy of $\mathbf{R}P^3$ disjoint from the first copy. The interval I is attached to the first copy at its first endpoint and to the second copy at its second endpoint. The open interval is disjoint from both copies of $\mathbf{R}P^3$.
a. What is $\pi_1(X, \star)$?
b. Describe the universal covering $\widetilde{X}$ of X.
c. Compute the homology of $\widetilde{X}$.

2. Let $f : M^n \to N^n$ be a smooth immersion of a connected smooth compact boundaryless n-manifold M into a smooth and connected n-manifold N.
a. Show that f is a covering projection.
b. Show that $S^2 \times S^2$ cannot be immersed in $\mathbf{R}^4$.

3. Let X be a Hausdorff space and suppose that each point has a neighborhood basis of simultaneously open and compact neighborhoods. Show that X is locally compact, and each connected component of X is a point.

4. Let S_g denote the oriented surface of genus g. If $f : S_g \to S_g$ is a map which is homotopic to the identity, show that if $g \neq 1$ then f must have a fixed point. Also, show that when $g = 1$, f can be chosen to have no fixed points.

5. Define

$$X_a = \{(x, y, z) \in \mathbf{R}^3 \ : \ x^2 - y^2 = az\},$$
$$Y_b = \{(x, y, z) \in \mathbf{R}^3 \ : \ x^2 + by = 1\}.$$

a. For which values of a is X_a a smooth submanifold of dimension 2 in $\mathbf{R}^3$?
b. For which values of b is Y_b a smooth submanifold of dimension 2 in $\mathbf{R}^3$?
c. For which values of a and b are X_a and Y_b smooth submanifolds of dimension 2 in $\mathbf{R}^3$ that intersect transversely in $\mathbf{R}^3$?