

Morning :

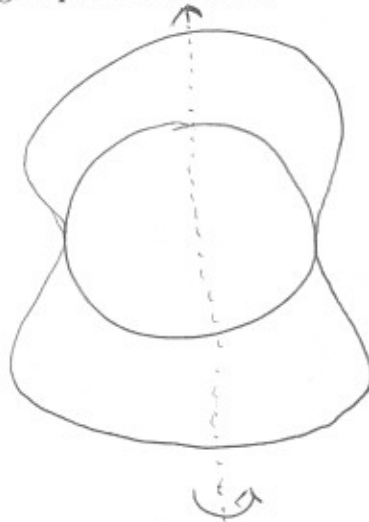
1. Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. A map  $\pi : X \rightarrow Y$  is called a *submetry* if for every  $x \in X$ , and any  $r > 0$ ,

$$\pi(D(x, r)) = D(\pi(x), r)$$

where  $D(x, r)$  denotes the closed  $r$ -ball about  $x$ .

- Show that  $\pi$  is surjective if  $X$  is nonempty.
- Show that  $\pi$  is continuous.
- Show that  $\pi$  is open. [A map  $f : A \rightarrow B$  is open if and only if for every open subset  $U \subset A$ , the image  $f(U)$  is open in  $B$ .]
- Suppose that  $y_1, y_2 \in Y$ . Suppose that  $x_1 \in X$  satisfies  $\pi(x_1) = y_1$ . Show that there exists  $x_2 \in X$  such that  $\pi(x_2) = y_2$  and  $d_X(x_1, x_2) = d_Y(y_1, y_2)$ .

2. The figure below is rotated about the  $z$ -axis to generate a topological space  $X$  that is the union of two 2-spheres joined at their common equator. Since  $X$  is symmetric about the origin we may form the quotient space  $Y = X / \sim$  of  $X$  by the antipodal map, that is,  $p \sim -p$ . Find the homology groups of  $X$  and  $Y$ .



3. Let  $U_1 \subset U_2 \subset \dots$  be a sequence of simply-connected open subsets of a topological space  $X$ . Show that  $U = \bigcup_{i=1}^{\infty} U_i$  is simply-connected.

4. Let  $f : S^2 \rightarrow \mathbb{R}^4$  be a smooth map which does not contain the origin. Show that there is a line through the origin in  $\mathbb{R}^4$  which is disjoint from  $f(S^2)$ .

5. Put  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ . Show that the one-point-compactification of  $\mathbb{R}_+^2$  is homeomorphic to a familiar space.

**UNIVERSITY OF MICHIGAN**  
**DEPARTMENT OF MATHEMATICS**  
**Qualifying Review Examination in Topology**  
*5 May 2006: Afternoon Session, 2:00-5:00*

1. Let  $X \subset \mathbb{R}^3$  be given as the union of the 2-spheres

$$S_1 = \{(x, y, z) : (x - 2)^2 + y^2 + z^2 = 1\},$$

$$S_2 = \{(x, y, z) : (x + 2)^2 + y^2 + z^2 = 1\}$$

and the line segments

$$L_1 = \{(x, 0, 1) : -2 \leq x \leq 2\}$$

and

$$L_2 = \{(x, 0, -1), -2 \leq x \leq 2\}.$$

Let  $Y$  be the quotient space of  $X$  under the map  $T$  which is the antipodal map on the spheres and interchanges the line segments, i.e.,

$$T(x, y, z) = \begin{cases} (4 - x, -y, -z), & (x, y, z) \in S_1, \\ (-4 - x, -y, -z), & (x, y, z) \in S_2, \\ (x, 0, -1), & (x, y, z) \in L_1, \\ (x, 0, 1), & (x, y, z) \in L_2. \end{cases}$$

Let  $p : X \rightarrow Y$  denote the quotient map.

- a) Find  $\pi_1(X)$ . Take  $(0, 0, 1)$  as base point.
- b) Find  $\pi_1(Y)$ . Take  $p(0, 0, 1)$  as base point.
- c) What is the image of  $\pi_1(X)$  in  $\pi_1(Y)$ ?
- d) Find the universal cover of  $Y$ .

2. Let  $A$  denote the space of  $2 \times 2$  real matrices with determinant 1 and let  $B$  denote the space of  $2 \times 2$  matrices with trace 0.

- a) Show that  $A$  is a submanifold of  $\mathbb{R}^4$ .
- b) Show that  $B$  is a submanifold of  $\mathbb{R}^4$ .
- c) Is  $A \cap B$  a submanifold of  $\mathbb{R}^4$ ? Explain.

3. Prove or disprove the following :

- (a) The continuous image of a locally connected space is locally connected.
- (b) The quotient of a locally connected space is locally connected.

4. Let  $X$  be a closed disk in  $\mathbb{R}^2$  with  $k$  disjoint open balls removed,  $k \geq 0$ . For which  $k$  is it true that any continuous map  $f : X \rightarrow X$  which is homotopic to the identity necessarily has a fixed point?

5. Notation: If  $A \subset X$  we denote by  $\partial A$  the boundary of  $A$  in  $X$ , i.e.  $\partial A = \overline{A} \cap \overline{X - A}$ . If  $A \subset X$  and  $B \subset Y$  and  $A$  and  $B$  are both closed prove that  $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$ .