UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology
2 September 2006: Morning Session, 9:00-12:00

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that $f^{-1}([-n, n])$ is compact for each positive integer n. Show that f achieves either a minimum value or a maximum value.

2. Construct a space X with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_3$, $H_2(X) = \mathbb{Z}$ and all other homology groups of X vanishing.

3. Let M be a smooth simply-connected compact manifold of dimension n. Is there an immersion of M into the n-torus $T^n = S^1 \times \ldots \times S^1$?

4. Put

$$\mathcal{O}_1 = \{U \subset \mathbb{R}^2 : p_1(U) \text{ is open in } \mathbb{R} \text{ and } p_2(U) \text{ is open in } \mathbb{R}\},$$

where $p_1(x, y) = x$ and $p_2(x, y) = y$.

Put

$$\mathcal{O}_2 = \{U \subset \mathbb{R}^2 : U \cap p_1^{-1}(x) \text{ is open in } p_1^{-1}(x) \text{ and } U \cap p_2^{-1}(y) \text{ is open in } p_2^{-1}(y), \text{ for each } x, y \in \mathbb{R}\}.$$

a) Is $\mathcal{O}_1$ a topology for $\mathbb{R}^2$? Explain carefully.
b) Is $\mathcal{O}_2$ a topology for $\mathbb{R}^2$? Explain carefully.

5. Let M^m be a smooth connected m-manifold (without boundary) whose fundamental group is finite of odd order. If $m < n$ show that any continuous map $f : M^m \to \mathbb{R}P^n$ is null-homotopic, that is, homotopic to a constant map.

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2 September 2006: Afternoon Session, 2:00-5:00

1. Let Z be the complement of the x-axis and the y-axis in $\mathbb{R}^3$, i.e.

$$Z = \mathbb{R}^3 - (\text{x-axis} \ \cup \ \text{y-axis}).$$

Compute the homology of Z.

2. Let $X = \mathbb{R}P^2 - point$. Compute the homology of X.

3. Let $f : M^m \to N^m$ be a smooth bijection so that $df : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. Show that f is a diffeomorphism.

4. Let $\mathbb{R}P^2$ be the real projective plane. Find all of the connected covering spaces of
(a) $\mathbb{R}P^2 \vee \mathbb{R}P^2$, the one-point union of two copies of $\mathbb{R}P^2$.
(b) $\mathbb{R}P^2 \# \mathbb{R}P^2$, the connected sum of two copies of $\mathbb{R}P^2$ (also known as the Klein bottle).

5. Let X be a compact space and $f : X \to Y$ a continuous map to a Hausdorff space Y. Show that the induced map $f : X \to f(X)$ is a quotient map.