UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

3 May 2007: Morning Session, 9:00-12:00

(AM1) Let p, q and r be positive prime integers and let a, b, c, and N be positive integers.

(a) Show that if $p > N$ then the Sylow p-subgroup of a group of order $p^a N$ is normal.

(b) Let G have order $p^a q^b r^c$ where $p > q^b r^c$ and $q > r^c$. Prove that G is solvable.

(AM2) Let M be the vector space of 3×3 matrices over the real numbers $\mathbf{R}$.

(a) Prove that there is a well-defined $\mathbf{R}$-linear map $T : \bigwedge^2 M \to M$ such that if $A, B \in M$ then $T(A \wedge B) = AB - BA$.

(b) Give a simple characterization of the matrices in the image of T.

(c) What is the dimension of the kernel of T?

(AM3) Let $n \geq 2$ be an integer. The classical adjoint of an $n \times n$ matrix $A = (a_{ij})$ is $\mathrm{adj}(A)$, whose i, j entry is $(-1)^{i+j} A_{ji}$, where A_{rs} is the determinant of the submatrix obtained from A by striking row r and column s. Suppose that A is an $n \times n$ matrix of integers with fundamental invariants $d_1, \ldots, d_n$, so that $d_i | d_{i+1}$, for $i = 1, \ldots, n-1$ and the cokernel of $A : \mathbf{Z}^n \to \mathbf{Z}^n$ is the direct sum of the modules $\mathbf{Z}/d_i\mathbf{Z}$. Assume that all d_i are nonzero. Let $d := \det(A) = \pm d_1 \cdots d_n$. Prove that every fundamental invariant of the integer matrix $\mathrm{adj}(A)$ is divisible by $\frac{d}{d_n} = \pm d_1 \ldots d_{n-1}$.

(AM4) Let $K \subseteq L$ be a field extension, where L is the splitting field of a separable polynomial $f(x) \in K[x]$ of degree 4. Let r_1, r_2, r_3 and r_4 be the roots of f in L. The Galois group G of L over K may be viewed as a subgroup of the permutations of r_1, r_2, r_3 and r_4. Suppose that G contains $\alpha = (r_1\ r_2\ r_3\ r_4)$, $\beta = (r_1\ r_2)(r_3\ r_4)$, and $\gamma = (r_1\ r_2\ r_3)$, in cycle notation for permutations.

(a) Given this information, must f be irreducible? Explain.

(b) Prove that G is isomorphic with the symmetric group on four elements.

(c) Prove that α and β generate a Sylow 2-subgroup H of G. What is $d = [L^H : K]$? How may fields K' with $K \subseteq K' \subseteq L$ are there such that $[K' : K] = d$?

(AM5) Let V be the real vector space of polynomials in a variable x of degree at most 2. Let $B : V \times V \to \mathbf{R}$ be the real symmetric bilinear form whose value on $(f(x), g(x))$ is the value of the derivative of $xf(x)g(x)$ at $x = 1$. E.g., $B(x-1, x+1)$ is the value of the derivative of $x(x-1)(x+1) = x^3 - x$, that is of $3x^2 - 1$, at $x = 1$, which is $3 - 1 = 2$.

(a) Find the matrix M associated with B with respect to a suitable basis for V.

(b) Find a diagonal matrix of the form $A^{\mathrm{tr}} M A$, where A is a real invertible matrix, and $^{\mathrm{tr}}$ indicates transpose.

(c) What are the rank and signature of B? Is B positive semi-definite? Is B negative semi-definite?

UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

3 May 2007: Afternoon Session, 2:00-5:00

(PM1) Let $p > 3$ be a prime number. Let Sym_Ω be the group of permutations on the set $\Omega := \{0, 1, 2, 3, \ldots, p\}$, and consider $\alpha := (0, 1)(2, 3)(4, 5)\ldots(p-1, p) \in \mathrm{Sym}_\Omega$ and any p-cycle $\beta \in \mathrm{Sym}_\Omega$. Let G be the subgroup of Sym_Ω generated by α and β.

(a) Prove that G acts transitively on the set $\{(a, b) \in \Omega \times \Omega : a \neq b\}$, where $g(a, b) = (g(a), g(b))$.

(b) Prove that G is not simple if $p \equiv 1$ modulo 4.

(c) Show that $|G|$ is divisible by $p(p + 1)$.

(PM2) Let q be a power of a prime integer p, and let $\mathbf{F}_q$ denote the finite field with q elements. The degree n extension $\mathbf{F}_{q^n}$ of $\mathbf{F}_q$ has the norm map $N : \mathbf{F}_{q^n} \to \mathbf{F}_q$, where $N(x) = x^{1+q+\cdots+q^{n-1}}$.

(a) Prove that N is onto.

(b) Prove that the set $S := \{x \in \mathbf{F}_{q^n} | N(x) = 1\}$ spans $\mathbf{F}_{q^n}$ as a vector space over its subfield $\mathbf{F}_q$.

(PM3) Let V be a finite-dimensional vector space over the complex numbers $\mathbf{C}$. Suppose that R is a commutative subring of $\mathrm{End}_{\mathbf{C}}(V)$ and that every element of R is semisimple (i.e., diagonalizable). Prove that V has a basis each of whose elements is an eigenvector of every element of R.

(PM4) Let R be a principal ideal domain.

(a) Let a and b be elements of R. Prove that the R-module of R-module homomorphisms from R/aR to R/bR is 0 if $a \neq 0$ and $b = 0$ and is isomorphic with $R/(a, b)R$ otherwise. In each case, describe explicitly a homomorphism that generates this cyclic module by specifying its value on the image of 1 in R/aR.

(b) Suppose that $R = K[x]$ is the polynomial ring in one variable over a field of characteristic 0, $M = R \oplus R \oplus R/(x^2 - 1)$, and $N = R \oplus R \oplus R \oplus R/(x - 1)^3$. Let H be the R-module of R-linear maps from M to N. Let T be the torsion-submdodule of H. Describe T as a direct sum of cyclic modules, and describe H/T.

(PM5) Describe all possible Jordan forms for a 4×4 matrix A over the complex numbers $\mathbf{C}$ that satisfy both $A^6 = A^4$ and $A^4 + A^2 = 2A^3$.