UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
4 May 2007: Morning Session, 9:00-12:00

1. Suppose that f is analytic in a domain $D \subset \mathbb{C}$ and that all values of f lie in the disk of radius $R > 0$ centered at the origin.

(a) Prove that

$$|f'(z)| \le \frac{R}{d(z)}, \qquad \text{for all } \ z \in D,$$

where $d(z)$ is the distance from z to the boundary of D.

(b) Use the result of Part (a) to prove Liouville's theorem: Every bounded analytic function in the whole plane $\mathbb{C}$ is constant.

2. Let $E \subset [0,1]$ denote the set of numbers x that have *some* decimal expansion

$$x = 0.a_1 a_2 a_3 \ldots \qquad \text{with } a_n \neq 7 \ \text{ for all } n\,.$$

Show that E is a measurable set, and calculate its measure. Explain your reasoning.

3. Let μ and ν be finite (positive) measures on a measurable space $(X, \mathcal{M})$, and suppose that

$$\nu(E) = \int_E f\, d\mu\,, \qquad \text{for all } E \in \mathcal{M}\,,$$

where f is some function in $L^1(\mu)$. Show that

$$\int_X g\, d\nu = \int_X gf\, d\mu \qquad \text{for all } g \in L^1(\nu)\,.$$

4. Construct a conformal mapping f of the domain

$$D_1 = \{z \in \mathbb{C} : |z| < 1\,,\ -\pi < \arg\{z\} < \pi\}$$

onto the domain $D_2 = \{z \in \mathbb{C} : 1 < \operatorname{Re}\{z\} < 2\}$. Draw figures to illustrate each step of the construction.

5. Does there exist an analytic function f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the properties $f(\mathbb{D}) \subset \mathbb{D}$, $f(\frac{1}{2}) = \frac{3}{4}$, and $f'(\frac{1}{2}) = \frac{2}{3}$? Justify your answer.

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6. Use the method of residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4}\, dx\,.$$

Show all estimates.

7. Let $f(x) \geq 0$ be continuous on the interval $[0, \infty)$, and suppose that $\int_0^\infty f(x)\, dx < \infty$. Prove that

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n x f(x)\, dx = 0\,.$$

8. Show that the function

$$f(x) = \int_0^{2\pi} \frac{20\, e^{5it} + 2x\, e^{2it}}{4\, e^{5it} + x\, e^{2it} + x^2}\, dt\,, \qquad 0 \leq x \leq 1\,,$$

is constant on the interval $[0, 1]$, and calculate its value.

9. Let $f(z) = z + z^2$ and let

$$V = \left\{ z \in \mathbb{C} : |z| < \frac{1}{2}\,,\ \frac{3\pi}{4} < \arg\{z\} < \frac{5\pi}{4} \right\}.$$

(a) Show that $f(V) \subset V$.

(b) Let f_n be the nth iterate of f. Thus $f_1(z) = f(z)$ and $f_{n+1}(z) = f(f_n(z))$ for $n = 1, 2, \ldots$. For each point $z \in V$, show that $f_n(z) \to 0$ as $n \to \infty$.

10. Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathcal{M}, \mu)$, and suppose that

$$\sum_{n=1}^{\infty} \mu\big(\{x \in X : |f_n(x)| > 1\}\big) < \infty\,.$$

Prove that

$$-1 \leq \liminf_{n \to \infty} f_n(x) \leq \limsup_{n \to \infty} f_n(x) \leq 1$$

for μ-almost every $x \in X$.