UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
8 September 2007: Morning Session, 9:00–12:00

1. Use the method of residues to calculate the integral

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx\,.$$

Show all estimates.

2. Let $E \subset \mathbb{R}$ be a compact (*i.e.*, closed bounded) set of real numbers. Suppose $\{f_n\}$ is a sequence of real-valued continuous functions which converges pointwise on E to a function f that is also continuous on E. Suppose further that the sequence $\{f_n\}$ is monotonic: $f_{n+1}(x) \le f_n(x)$ for all $x \in E$ and all $n = 1, 2, \ldots$.

(a) Prove that $f_n(x) \to f(x)$ *uniformly* on E as $n \to \infty$.

(b) Show by example that the hypothesis of compactness is essential.

3. Let μ and ν be finite positive measures on a σ-algebra $\mathcal{M}$ of subsets of some set X. Prove that the following two statements are equivalent.

(a) ν is absolutely continuous with respect to μ; *i.e.*, $\nu(A) = 0$ for every set $A \in \mathcal{M}$ such that $\mu(A) = 0$.

(b) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\nu(A) < \varepsilon$ for every set $A \in \mathcal{M}$ with $\mu(A) < \delta$.

4. Suppose that f is a bounded analytic function on the domain $\{z \in \mathbb{C} : |z| > 1\}$.

(a) Prove that $\lim_{z\to\infty} f(z)$ exists.

(b) Let L denote the limit in (a), and let Γ_R denote a circle $|\zeta| = R > 1$. Show that

$$\frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) - L\,, \qquad |z| > R\,,$$

where the contour Γ_R is traversed in the *clockwise* direction.

5. (*a*) Let $\mathbb{D}$ denote the unit disk. Is there an analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 1/2$ and $f'(0) = 3/4$? Either find such a function f or explain why it does not exist.

(*b*) Answer the same question for $f(0) = 1/2$ and $f'(0) = 4/5$.

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6. Suppose f and g are nonnegative measurable functions on the interval $[0,1]$, with the properties

$$\int_0^1 f(x)\,dx = 2\,, \qquad \int_0^1 g(x)\,dx = 1\,, \qquad \text{and} \quad \int_0^1 [f(x)]^2\,dx \le C$$

for some constant $C > 4$. Let $E = \{x \in [0,1] : f(x) > g(x)\}$. Show that E has measure $m(E) \ge 1/C$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Prove carefully that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|\,dx = 0\,.$$

8. Let Ω be the domain consisting of the unit disk $\mathbb{D}$ with the real segment $[\frac{1}{2}, 1)$ removed. Construct a conformal mapping from Ω onto $\mathbb{D}$. Draw diagrams to illustrate each step of the mapping.

9. Suppose that $f : [0,1] \to [0,\infty]$ is in L^1 and that

$$\int_E f(x)\,dx \le \sqrt{m(E)}$$

for every measurable set $E \subset [0,1]$, where m denotes Lebesgue measure. Show that $f \in L^p$ for every $p < 2$, but that f need not belong to L^2.
Hint. Consider $A_n = \left\{x \in [0,1] : 2^n \le f(x) < 2^{n+1}\right\}$.

10. Let $f(z) = |z|^{2k} + \lambda\,\overline{z}^{\,2k}$, where λ is a complex constant with $|\lambda| \ne \frac{1}{2}$ and k is a positive integer. Find the winding number of the curve $w = (\partial f/\partial \overline{z})(z)$ about the origin as z traverses the unit circle in the positive (counterclockwise) direction. Explain your reasoning.

Definitions: $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$