

UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Topology

5 May 2007: Morning Session, 9:00 -12:00

1. Let  $f: X \rightarrow Y$  be a continuous map of metric spaces. The limit set  $L(f)$  of  $f$  is the set of  $y$  in  $Y$  such that  $y = \lim f(x_n)$  for some sequence  $(x_1, x_2, \dots)$  in  $X$  with **no** convergent subsequence.
  - a) Show that  $f(X)$  is closed in  $Y$  if and only if  $L(f) \subset f(X)$ .
  - b) Give an example of a continuous  $f: \mathbb{R} \rightarrow \mathbb{R}^2$  with  $f(\mathbb{R})$  closed and  $L(f) \neq \emptyset$ .
2. Let  $X$  be the complement of two circles  $\{x^2 + y^2 = 1, z = 1\}$  and  $\{x^2 + y^2 = 1, z = -1\}$  in  $\mathbb{R}^3$ . Show that  $X$  is path connected and determine  $\pi_1(X)$  and  $H_*(X)$ .
3. Let  $f: S^2 \rightarrow \mathbb{R}$  be given by  $f(x, y, z) = xy + z^2$ . Find the critical points of  $f$ .
4. Let  $K$  be the simplicial complex consisting of the 3-simplices  $(v_1, v_2, v_3, v_4)$ ,  $(v_3, v_4, v_5, v_6)$ ,  $(v_1, v_2, v_5, v_6)$  and their faces. (All the  $v_k$  are distinct.) Compute  $H_*(K)$ .
5. Prove or disprove:
  - a) The space  $X$  has the discrete topology if and only if for each  $x$  in  $X$ , the connected component of  $x = \{x\}$ .
  - b)  $\prod_{n=1}^{\infty} X_n$  has the discrete topology if and only if  $X_n$  has the discrete topology for each  $n$ .

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5 May 2007: *Afternoon Session, 2:00 -5:00*

1. a) Let  $\Delta \subset S^n \times S^n$  be the diagonal subspace;  $\Delta = \{(x, x) \mid x \in S^n\}$ . Show that the projection  $p : S^n \times S^n - \Delta \rightarrow S^n$  given by  $p(x, y) \rightarrow x$  is a homotopy equivalence.  
  
b) Let  $T^3 = S^1 \times S^1 \times S^1$  and let  $\Delta \subset T^3 \times T^3$  be the diagonal subspace.  
Compute  $\pi_1(T^3 \times T^3 - \Delta)$ .  
  
c) Show that the projection  $p : T^3 \times T^3 - \Delta \rightarrow T^3$  given by  $p(x, y) \rightarrow x$  is **not** a homotopy equivalence.
  
2. a) Let  $A$  be a bounded subset of  $\mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}$  a uniformly continuous map. Must  $f$  be bounded? Give a counterexample or a careful proof .  
  
b) Let  $A$  be a bounded subset of  $l_2$  and  $f : A \rightarrow \mathbb{R}$  a uniformly continuous map. Must  $f$  be bounded? Give a counterexample or a careful proof .  
  
(Recall that  $l_2 = \{(x_1, x_2, \dots) \mid \sum_{n=1}^{\infty} x_n^2 < \infty\}$ ;  $l_2$  is endowed with the metric  
$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$$
 .)
  
3. Describe the set of homotopy classes of maps from  $(S^1 \times S^2, p)$  to  $(S^1, 1)$  and prove your answer. Recall that  $S^1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$ . Here  $p = (1, *)$  for some point  $*$  in  $S^2$ .
  
4. Let  $X_q = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 - y^2 + w = q, z^2 + w^2 = 1\}$ . For what values of  $q$  is  $X_q$  a manifold?
  
5. Let  $X = (S^1 \times S^1)/\sim$  where  $(x, y) \sim (y, x)$ . Compute  $H_*(X)$ .