UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

2 January 2008: Morning Session, 9:00-12:00

Problem 1. Let G be a finite group and H a subgroup of G. Let $n = [G\colon H]$ be the index of H in G.

a) Show that $[G\colon \cap_{x\in G} xHx^{-1}]$ divides $n!$.
b) Suppose that $[G\colon H]$ is the minimal prime factor of $|G|$. Show that H is a normal subgroup.

Problem 2. Let f be an irreducible polynomial in $\mathbf{Q}[x]$, of degree n. Suppose that the Galois group G of f is abelian, of order m.

a) Show that $m = n$.
b) Must G be cyclic ? Justify your answer.

Problem 3. Let K be a subfield of $\mathbf{C}$ and A an $n \times n$ matrix with coefficients in K.

a) Show that the subalgebra $K[A]$ of $M_n(K)$ generated by A is a direct product of fields if and only if the minimal polynomial of A is reduced.
b) Deduce that A is diagonalizable over $\mathbf{C}$ if and only if $K[A]$ is a direct product of fields.

Problem 4. Suppose a finite group G acts on a finite set X. For $g \in G$, let $X^g = \{x \in X \mid g \cdot x = x\}$, and for $x \in X$, let $G_x = \{g \in G \mid g \cdot x = x\}$. Prove that

$$\sum_{g\in G} |X^g| = \sum_{x\in X} |G_x|,$$

where $|\cdot|$ denotes cardinality. Show that this number is equal to $|G|$ times the number of orbits.

Problem 5. Let V be a vector space over a field F.

a) Show that there is a unique linear map $T\colon V \otimes_F V \otimes_F V \to V \otimes_F V \otimes_F V$ such that $T(u \otimes v \otimes w) = w \otimes u \otimes v$ for all u, v, and $w \in V$.
b) If V has finite dimension n, find the minimal and characteristic polynomials of T.

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2 January 2008: Afternoon Session, 2:00-5:00

Problem 1. Let M be the cokernel of the mapping from $\mathbf{Z}^2$ to $\mathbf{Z}^3$ given by the matrix

$$\begin{pmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{pmatrix}$$

a) Write M as a direct sum of cyclic abelian groups.
b) How many homomorphisms are there from M to $\mathbf{Z}/3\mathbf{Z}$?
c) Write $M \otimes_{\mathbf{Z}} M$ as a direct sum of cyclic groups.

Problem 2. Let G be a finite group, P a Sylow p-subgroup of G, and $N = N_G(P)$.

a) Show that if S is a subgroup of G that contains N, then $N_G(S) = S$.
b) Show that if $S \subseteq T$ are subgroups of G that contain N, then $[T\colon S] \equiv 1 \pmod{p}$.

Problem 3. Let K be a field of characteristic $p > 0$, and let $f = x^{2p} - tx^p + t \in K(t)[x]$.

a) Show that f is irreducible over $K(t)$.
b) If L is the splitting field of f, what is $[L\colon K(t)]$?

Problem 4. Let K be a field, and $F = \sum_{i+j>0} a_{i,j} X^i Y^j$ a power series in $K[[X,Y]]$, and

$$R = K[[X,Y]]/(F).$$

Show that the following are equivalent:

a) R is a discrete valuation ring.
b) R is a principal ideal domain.
c) $a_{1,0} \neq 0$ or $a_{0,1} \neq 0$.

Problem 5. Let k be a field with $\mathrm{char}(k) \neq 2$, and V a finite dimensional vector subspace over k. Let ω be a *symplectic form* on V, that is, an alternating bilinear form $V \times V \to k$. Suppose that ω is nondegenerate as a bilinear form. Given a linear subspace W of V, we put

$$W^\perp := \{u \in V \mid \omega(u,w) = 0 \text{ for every } w \in W\}.$$

a) Show that $\dim(V)$ is even.
b) A vector subspace W of V is called *coisotropic* if $W^\perp \subseteq W$. Show that every hyperplane in V is coisotropic.
c) Show that if W is a coisotropic subspace of V, then $\dim(W) \geq \frac{1}{2}\dim(V)$.