UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

2 May 2008: Morning Session, 9:00-12:00

Problem 1. Let G be a group of order a positive power of a prime p.

a) If G acts on a finite set X, show that the number of fixed points is congruent to the cardinality of X modulo p.
b) Prove that G has a nontrivial center.

Problem 2. Let k be a field, and suppose that $E \supset k$ is an integral domain that is finite dimensional as a vector space over k.

i) Explain why E is a field.
ii) If $a \in E$, let α_a be the k-linear transformation $E \to E$ defined by $\alpha_a(x) = ax$, and let $N(a) = \det(\alpha_a)$. Show that N gives a multiplicative map $E^* \to k^*$, where E^* and k^* denote the set of nonzero elements in E and k, respectively.
iii) If $a \in E$ is purely inseparable over k, that is, if its minimal polynomial has the form $f(T) = T^{p^e} - r$, where $r \in k$, $p = char(k) > 0$, describe the Jordan canonical form of α_a over the algebraic closure of k.

Problem 3. Let V be a finite-dimensional vector space over a field F, with $\dim_F(V) = n$. Fix a positive integer $p \leq n-1$.

i) Show that there is a nonzero canonical linear map
$$\phi\colon \wedge^p V \otimes V \to \wedge^{p+1} V$$
(*canonical* means *independent of choices of bases*).
ii) Is ϕ surjective ? Injective ? Justify!
iii) Show that if $u \in \wedge^p V$ is such that $\phi(u \otimes v) = 0$ for every $v \in V$, then $u = 0$.

Problem 4. Let $L = F(x, y)$, with F a field of characteristic $p > 0$, and x and y indeterminate. Let $K = F(x^p, y^p)$.

a) Show that the degree $[L : K]$ of the field extension is p^2.
b) Is there a u in L with $L = K(u)$? Why or why not?
c) Is the number of subfields of L containing K finite or infinite? Why?

Problem 5. Let K be the splitting field over $\mathbf{Q}$ of an irreducible cubic polynomial $f(x)$. Prove that if $f(x)$ has exactly one real root, then $[K\colon \mathbf{Q}] = 6$, and that the Galois group of $f(x)$ is isomorphic to S_3.

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Qualifying Review Examination in Algebra

2 May 2008: Afternoon Session, 2:00-5:00

Problem 1. Let A be the algebra $\mathbf{C}[x]/((x^2-2)^3)$, with x an indeterminate. Let $L\colon A \to A$ be the complex linear map given by multiplication by x^3.

a) Find the minimal polynomial of L.
b) Find the Jordan canonical form of L.

Problem 2. Let G be a group whose automorphism group $\mathrm{Aut}(G)$ is cyclic. Prove that G is commutative.

Problem 3. Given a field K, for which a, b, and c in K is there a 3 by 3 matrix A with entries in K such that $\det(A) = a$, $\mathrm{tr}(A) = b$, and $\mathrm{tr}(A^2) = c$? (Your answer may depend on the characteristic of K.)

Problem 4. Let $\zeta = e^{\pi i/6}$ $(= \exp(\pi\sqrt{-1}/6)$.

a) Find the minimal (monic) polynomial of ζ over $\mathbf{Q}$.
b) Find the Galois group of $\mathbf{Q}(\zeta)$ over $\mathbf{Q}$.
c) Write each subfield of $\mathbf{Q}(\zeta)$ in the form $\mathbf{Q}(u)$ (for an explicit u).

Problem 5. Let $C([0,1])$ be the ring of continuous functions on the unit interval $[0,1]$, with values in $\mathbf{R}$. For each point $p \in [0,1]$, set $\mathfrak{m}_p = \{f | f(p) = 0\}$.

i) Show that $\mathfrak{m}_p$ is a maximal ideal of $C([0,1])$.
ii) Prove that every maximal ideal of $C([0,1])$ is equal to some $\mathfrak{m}_p$ (Hint: use the fact that $[0,1]$ is compact).