UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

6 September 2008: Morning Session, 9:00-12:00

Problem 1. Let $G \subset O(3)$ be the subgroup of the orthogonal group mapping the set of eight vertices $(\pm 1, \pm 1, \pm 1)$ of a cube into itself. Find the order of G, and, for each prime p dividing the order, find a p-Sylow subgroup of G.

Problem 2. Let V be a finite-dimensional vector space over a field, and let $T\colon V \to V$ be a linear map. Let $V_\infty := \bigcup_{n\geq 1} \operatorname{Ker}(T^n)$, and $V^\infty := \bigcap_{n\geq 1} \operatorname{Im}(T^n)$. Show that $V = V^\infty \oplus V_\infty$.

Problem 3. Let $\rho\colon G \to GL(V)$ be a homomorphism from a finite group G to the group of automorphisms of a finite-dimensional real vector space V.

i) Show that V has an inner product $(\cdot,\cdot)$ such that $(\rho(g)(v), \rho(g)(w)) = (v, w)$ for all $v, w \in V$ and $g \in G$.
ii) Give an example of an infinite abelian group G with such a ρ for which the assertion in i) is false.

Problem 4. Let R be a principal ideal domain, and $\phi\colon R^n \to R^n$ the linear map determined by an $n \times n$ matrix A with entries in R. Let M be the cokernel of ϕ, and D be the determinant of A.

i) Show that $D \cdot x = 0$ for all $x \in M$.
ii) Does this extend to the case where R is an arbitrary commutative ring ?

Problem 5. Let $E \subset F$ be a finite field extension.

i) Show that if the extension is normal, and if f is an irreducible polynomial in $E[x]$, then all the irreducible factors of f in $F[x]$ have the same degree.
ii) Show that conversely, if for every irreducible $f \in E[x]$, all the irreducible factors of f in $F[x]$ have the same degree, then the extension is normal.

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6 September 2008: Afternoon Session, 2:00-5:00

Problem 1. Let G be a finite group.

i) Show that if H is a subgroup of G, then the number of conjugates gHg^{-1}, with $g \in G$, is no larger than $[G\colon H]$.
ii) Use i) to show that if H is a proper subgroup of G, then G is not the union of the conjugates of H.
iii) Show that if p is a prime that divides $|G|$, and if $|G| \leq p^2$, then G contains a normal p-subgroup.

Problem 2. Let K be a field, G a finite group of automorphisms of K, and $F \subseteq K$ the fixed field of G. For $x \in K$, let $\{x_1, \ldots, x_n\}$ be the orbit of x with respect to the action of G. Show that $\prod_{i=1}^n (X - x_i)$ is the irreducible polynomial of x over F.

Problem 3. Let $L\colon V \to V$ be a linear transformation of a finite-dimensional vector space, with characteristic polynomial $X^n - a_1X^{n-1} + a_2X^{n-1} - \cdots + (-1)^n a_n$. Show that a_k is the trace of $\wedge^k L\colon \wedge^k V \to \wedge^k V$.

Problem 4.

i) Find the minimal polynomial of $\sqrt{4+\sqrt{7}}$ over $\mathbf{Q}$.
ii) Find the Galois group of that polynomial's splitting field over $\mathbf{Q}$. (Hint: show that $\sqrt{4+\sqrt{7}} = \frac{\sqrt{2}+\sqrt{14}}{2}$).

Problem 5. Let F be a finitely generated free module over the principal ideal domain R, and let $(e_\lambda)_{\lambda\in\Lambda}$ be a basis of F.

i) Given $a \in F$, with $a = \sum_{\lambda\in\Lambda} a_\lambda e_\lambda$, show that the ideal $\sum_{\lambda\in\Lambda} a_\lambda R$ depends only on a, and not on the given basis.
ii) The element $a \in F$ is called primitive if $\sum_{\lambda\in\Lambda} a_\lambda R = R$. Show that a is primitive if and only if it occurs in some basis of F.