UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
5 January 2008: Morning Session, 9:00-12:00

1. Use the method of residues to calculate the integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

Show all estimates.

2. Assume that $f \in L^\infty(E)$, where $m(E) < \infty$. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p(E)} = \|f\|_{L^\infty(E)}.$$

3. What is the value of

$$\int_c \frac{32z^7 + \frac{3z^2}{2\sqrt{z^3+2}}}{4z^8 + \sqrt{z^3+2}}\, dz,$$

where c is the positively oriented unit circle.

4. Assume that $f,\ f_k \in L^2(R^n)$, for $k = 1, 2, \ldots$ and

$$f_k \to f \quad \text{a.e. in } R^n, \quad \|f_k\|_{L^2(R^n)} \to \|f\|_{L^2(R^n)}, \quad \text{as } k \to \infty.$$

Show that f_k converge to f in $L^2(R^n)$.

5. True or false. Give a short explanation.

a. There exist a function f analytic in the unit disc D such that

$$f(0) = 0, \quad f(\frac{1}{2}) = \frac{1}{10}, \quad \text{and } Re\{f(z)\} \geq 0, \text{for all } z \in D.$$

b. One can find a function g that is analytic in $\{z \in C : 0 < |z| < \epsilon\}$, g has an essential singularity at 0 and $\dfrac{1}{g(z) - 1}$ is bounded.

c. If h is analytic in Ω and γ is a simple closed curve in Ω, then $\int_\gamma h(z)\, dz = 0$.

d. Suppose f is entire and $|f(z)| \leq 1 + |z|^{1/2}$ for all z, then f is a constant.

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Qualifying Review Examination in Analysis

5 January 2008: Afternoon Session, 2:00-5:00

1. Assume that $f \in C^1[0,\pi]$, and $\int_0^\pi f(x)\,dx = 0$. Show that there exists a constant C, independent of f, such that

$$\int_0^\pi |f(x)|^2\,dx \le C \int_0^\pi |f'(x)|^2\,dx.$$

2. Find a conformal map from $D = \{z \in C : |z| < 1\} \setminus \{iy : \frac{1}{3} \le y \le 1\}$ to $\Omega = \{z : 1 < Re\, z < 2\}$.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K(x,y)$ be measurable with respect to the product σ-algebra $\mathcal{M} \times \mathcal{N}$. Assume that there is a constant A, such that for all $x \in X$,

$$\int_Y |K(x,y)|\,d\nu(y) \le A,$$

and for all $y \in Y$,

$$\int_X |K(x,y)|\,d\mu(x) \le A.$$

Let $1 \le p \le \infty$, and for $f \in L^p(X, \mathcal{M}, \mu)$, define

$$T(f)(y) = \int_X K(x,y) f(x)\,d\mu(x).$$

Show that

$$\|T(f)\|_{L^p(\nu)} \le A \|f\|_{L^p(\mu)}.$$

4. a. Suppose Ω is a domain, D is a disc such that $\overline{D} \subset \Omega$, f is analytic in Ω, f is not constant and $|f|$ is constant on the boundary of D. Prove that f has at least one zero in D.

b. Find all entire functions f such that $|f(z)| = 1$ when $|z| = 1$.

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5. Let $f_n : R \to [0, \infty)$ be nondecreasing for each natural number n. Assume that for all $x \in R$,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) < \infty.$$

Show that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad a.e.\ x \in R.$$