

UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
3 May 2008: Morning Session, 9:00–12:00

1. True or false. Give a short explanation.

Let $\Delta = \{z : |z| < 1\}$ be the unit disc, C be the complex plane.

- a. There exists an analytic function f on Δ such that $f(0) = 0$ and $\Re\{f(z)\} > 0$ when $z \neq 0$.
- b. There exist an analytic map $f : \Delta \rightarrow \Delta$ such that $f(0) = 0$, $f'(0) = \frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}$ and $f(\frac{1}{2}) = \frac{3}{4}$.
- c. We can find a non-constant entire map Φ such that $\Phi(C) \subset \{z : -\pi < \arg z < \pi, z \neq 0\}$.
- d. Let $u(x, y) = x^2 + xy$, then u is the real part of an analytic function.

2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be absolutely continuous, satisfy $f(0) = 0$ and $f' \in L^2([0, 1])$. Show that

$$\lim_{x \rightarrow 0^+} x^{-1/2} f(x)$$

exists and determine the value of this limit.

3. Find a conformal map from D to Ω where $D = \{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0, \text{ and } |z| < 1\}$ and $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$.

4. Let n be a positive integer and $a > e$ be a real number. Show that the equation $e^z = az^n$ has n solutions inside the unit disc.

5. (Generalized Minkowski inequality) Let $1 \leq p < \infty$. Assume that $f = f(x, y)$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$, and for a.e. $y \in \mathbb{R}^n$, $f(\cdot, y) \in L^p(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x, y)|^p dx \right]^{1/p} dy < \infty.$$

Show that

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x, y) dy \right|^p dx \right]^{1/p} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x, y)|^p dx \right]^{1/p} dy.$$

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3 May 2008: Afternoon Session, 2:00–5:00

6. Let (X, \mathcal{M}, μ) be a measure space and $f : X \rightarrow \mathbb{R}$ a measurable function. For each $\lambda > 0$, define

$$m(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}).$$

Show that for $1 \leq p < \infty$,

$$\int_X |f(x)|^p d\mu(x) = p \int_0^\infty \lambda^{p-1} m(\lambda) d\lambda.$$

7. Use the method of residues to calculate the integral

$$\int_0^{2\pi} \frac{\cos x}{2 + \cos x} dx$$

Show all steps.

8. Assume that $f = f(x, y)$ is a function defined on $E \times (a, b)$. For each fixed $y \in (a, b)$, f is integrable with respect to x on E , and for each fixed $x \in E$, f is differentiable with respect to y on (a, b) . Assume that there is a function $g \in L^1(E)$, such that

$$\left| \frac{d}{dy} f(x, y) \right| \leq g(x), \quad (x, y) \in E \times (a, b).$$

Show that $\int_E f(x, y) dx$ is differentiable with respect to y and

$$\frac{d}{dy} \int_E f(x, y) dx = \int_E \frac{d}{dy} f(x, y) dx.$$

9. Prove that if f is an analytic polynomial and one-to-one, then

$$f(z) = az + b$$

where a, b are constants, and $a \neq 0$.

10. Let $f \in L^1(\mathbb{R})$ and $f \geq 0$. Assume that for every $x \in \mathbb{R}$ there exists a sequence of intervals $I_{x,k}$ containing x , such that

$$|I_{x,k}| \rightarrow 0, \quad \text{and} \quad \frac{1}{|I_{x,k}|} \int_{I_{x,k}} f(y) dy \rightarrow 0$$

as $k \rightarrow \infty$. Show that $f = 0$ a.e.