UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
30 August 2008: Morning Session, 9:00-12:00

1. Compute
$$\int_0^\infty \frac{1}{1+x^n}\,dx, \qquad n = 2, 3, \ldots$$
using contours along the lines $\{z : \arg z = \pm\frac{2\pi}{n}\}$ and a segment of a circle. Do all estimates.

2. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $a_n \neq 0$. Use Rouché's theorem to prove that P has exactly n roots counting multiplicity in $\mathbf{C}$.

3. Let $f : [0,1] \to R$ be a measurable function. Show that there exists a unique $a_0 \in R$, such that
$$m(\{x \in [0,1] : f(x) \geq a_0\}) \geq 1/2, \qquad \text{and}$$
$$m(\{x \in [0,1] : f(x) \geq a\}) < 1/2, \qquad \text{for all } a > a_0.$$

4. Assume that f is analytic and one-to-one on $D = \{z : |z| < 1\}$ and $f(z) = z + z^2 g(z)$, where g is analytic in D. Prove that if $f(D) \subset D$ or $D \subset f(D)$, then $g(z) = 0$ for all $z \in D$.

5. 1. For $k = 1, 2, \ldots n$, let $R_k = R$, $f_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ be a nonnegative measurable function on $R_1 \times \cdots \times R_{k-1} \times R_{k+1} \times \cdots \times R_n$. Let
$$I_k = \int_{R^{n-1}} f_k^{n-1}\,dx_1 \ldots dx_{k-1}dx_{k+1} \ldots dx_n, \qquad k = 1, 2, \ldots n.$$
Show that
$$\int_{R^n} f_1 f_2 \ldots f_n\,dx_1 \ldots dx_n \leq (I_1 \ldots I_n)^{1/(n-1)}.$$

2. Let V be a bounded closed domain in R^3, S_1, S_2 and S_3 be the areas of the projections of V onto the three coordinate planes respectively. Show that
$$m(V) \leq \sqrt{S_1 S_2 S_3}.$$

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30 August 2008: Afternoon Session, 2:00-5:00

6. Assume that $E \subset R^n$ is Lebesgue measurable and $m(E) > 0$. Show that there exists a point $x \in E$, such that for every $\delta > 0$,

$$m(E \cap B(x, \delta)) > 0.$$

Here $B(x, \delta)$ is the ball centered at x with radius δ.

7. Show that $u(x, y) = x^4 + x^2 + e^{-y} \cos x - 6x^2 y^2 + y^4 - y^2$ is harmonic. Find an analytic function h such that $Re\{h\} = u$.

8. Assume that $f \in L([a, b])$ and

$$\int_a^b x^n f(x)\, dx = 0 \qquad \text{for all } n = 0, 1, 2, \ldots.$$

Show that $f(x) = 0$ a.e. $x \in [a, b]$.

9. Let

$$\Omega = \{z \in \mathbf{C} : Re\{z\} > 0, \text{ and } |z - 1| > 1\}.$$

Find a conformal map from Ω to the unit disc. (Hint: first map Ω into a strip.)

10. Let $1 < p < \infty$, $f_k \in L^p(E)$, $k = 1, 2, \ldots$, and

$$\lim_{k \to \infty} f_k(x) = f(x) \quad a.e., \qquad \sup_{1 \le k < \infty} \|f_k\|_p \le M.$$

Show that for any $g \in L^{p'}(E)$, where p' is the conjugate of p,

$$\lim_{k \to \infty} \int_E f_k(x) g(x)\, dx = \int_E f(x) g(x)\, dx.$$