UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Topology

1 May 2008: Morning Session, 9:00 -12:00

1) Let M be a simply connected manifold, let ~ be an equivalence relation on M and let X = M/~ be the quotient space. Suppose that i) every equivalence class has exactly 2 points and ii) for every point x in M there are open sets U(x), V(x) in M such that $x \in U(x)$, $U(x) \cap V(x) = \varnothing$ and for every u in U(x) there is a v in V(x) with u ~ v and for every v in V(x) there is a u in U(x) with v ~ u.

Let x_0 be a point in X and compute $\pi_1(X\ x_0)$. Can you describe a generator of $\pi_1(X, x_0)$?

2) Let X be a compact Hausdorff space and let $A_1, A_2, \ldots$ be a sequence of connected subspaces with $\overline{A_j} - A_j \subset A_{j+1}$ for each j = 1, 2, …. Suppose also that for each x in X there is an open set U(x) containing x such that $U(x) \cap A_j = \varnothing$ for all but a finite number of j. Prove that $\bigcup_{j=1}^{\infty} A_j$ is compact.

3) Let $f : X \to Y$ be a differentiable map of smooth compact simply connected n-manifolds. Show that f is a submersion if and only if f is a diffeomorphism.

4) Let F be a free group and R be a normal subgroup. Assume that F is finitely generated and R is finitely normally generated, and put π = F /R.

(a) Construct a finite 2-dimensional complex X such that $\pi_1 (X, x_0) = \pi$.

(b) If Y is a space such that $\pi_1 (Y , y_0) = \pi$, prove that there is a map $f : (X, x_0) \to (Y , y_0)$ such that $f_* : \pi_1 (X , x_0) \to \pi_1 (Y , y_0)$ is an isomorphism.

5) a) Compute the singular homology groups $H_*(S^n / S^k)$ where $S^k = R^{k+1} \cap S^n$ and S^n is the unit sphere in R^{n+1} .

b) Prove carefully that $H_*(S^n - S^k) = H_*(S^n - T)$ where T is a small tubular neighborhood of S^k in S^n. Note that T is diffeomorphic to $S^k \times D^{n-k}$.

c) Compute the singular homology groups $H_*(S^n - S^k)$.

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Qualifying Review Examination in Topology

1 May 2008: Afternoon Session, 2:00 -5:00

1) Let X be $\mathbb{R}^2$ with the standard topology τ_1 and let Y be $\mathbb{R}^2$ with the topology τ_2 given by: C is closed in Y if $C \cap L$ is closed in L(standard topology) for **every** straight line L in $\mathbb{R}^2$.

a) Show that, in fact, τ_2 is a topology on $\mathbb{R}^2$.

b) Are the topologies τ_1 and τ_2 comparable? Are they equal?

2) Consider two embedded circles in the solid torus $M = S^1 \times D^2 =$
$\{(x, y) \in R^2 \mid x^2 + y^2 = 1\} \times \{(x, y) \in R^2 \mid x^2 + y^2 \leq 1\}$;
$\alpha = S^1 \times \{(0, 0)\}$ and $\beta = \{\{(0, 1)\} \times \{(x, y) \in R^2 \mid x^2 + y^2 = 1/2\}$.

Let T_α , T_β be small open tubular neighborhood of α, β such that they are disjoint from each other and the boundary of M . Prove or disprove that $M - T_\alpha$, $M - T_\beta$ are homeomorphic.

3) Let X denote the union of two circles meeting at one point x_0 . Find a suitable connected 3-fold covering space of X and **use the cover** to prove that $\pi_1(X, x_0)$ is not abelian.

4) Prove that there is no submersion of a smooth nonempty compact manifold into Euclidean space.

5) Let M be the smooth manifold given by $M = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, z^2 + w^2 = 1\}$ and let $f : M \to \mathbb{R}$ be given by f(x,y,z,w) = x + z. Find all the critical points of f and the associated critical values.