UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

5 January 2009: Morning Session, 9:00-12:00

1. Let G be the group of invertible 4×4 matrices over the complex numbers, and let M be the set of all 4×4 complex matrices.

(a) Consider the action of $G \times G$ on M given by (g, h) acts on m by the matrix multiplication gmh^{-1}. Describe the orbits of this action.

(b) Consider the action of G on M by conjugation: g acts on m by the matrix multiplication gmg^{-1}. For what λ and μ are the two matrices below in the same orbit?

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

2. Consider the real quadratic form $Q(x, y, z) = x^2 + 4xy + y^2 + \lambda z^2$, where λ is some fixed real number. Compute the rank and signature of Q as a function of λ.

3. Let R be the ring $\dfrac{\mathbf{Z}[x, y]}{(12, x^2, y^3)}$. Find all prime ideals of R, giving explicit generators for each.

4. Fix a vector space V of dimension three over the finite field $\mathbf{F}_q$ of q elements. Find the cardinality of the following sets (with full justification):

(a) The set of all linear transformations $T : V \to V$.

(b) The set of all invertible linear transformations $T : V \to V$.

(c) The set of isomorphism classes of $\mathbf{F}_q[X]$-module structures on V.

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5. Consider the polynomial $F(x) = (x^2 - 2)(x^2 - 3)$ over $\mathbf{Q}$, and let E be its splitting field.

(a) Compute the degree of $E/\mathbf{Q}$ and find an element $\alpha \in E$ such that $E = \mathbf{Q}(\alpha)$.

(b) Describe the Galois group G of $E/\mathbf{Q}$, giving explicit generators and relations.

(c) Describe the lattice of subgroups of G, giving explicit generators and relations for each.

(d) Describe the lattice of intermediate fields K between $\mathbf{Q}$ and E, and explain the relation with part (c). Which of these field extensions are Galois?

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Qualifying Review Examination in Algebra

5 January 2009: Afternoon Session, 2:00-5:00

1. Let G be a group.

(a) Characterize what it means for G to be solvable, and what it means for G to be nilpotent.

(b) Given an example of a finite group that is solvable but not nilpotent.

(c) For each group given below, if G is the Galois group of the splitting field of a polynomial f over the rational numbers, can f be solved by radicals? Explain your answer.

1. $G = S_4 \times S_4 \times S_4$
2. $G = S_5$
3. G is the group of symmetries of a regular n-sided polygon, $n \geq 3$.

2. Let K be a field of 27 elements. Find the number of solutions of the equation $x^{13} = 1$ in K, and the number of solutions of the equation $x^{13} = -1$ in K.

3. Let G be a finite group of order $p^e m$, where p, e and m are postive integers, with p prime and not a divisor of m. Suppose that the Sylow p-subgroups of G are not normal. Show that there is a homomorphism θ of G to the symmetric group S_h on h elements, where $h > 1$ is a divisor of m that is congruent to 1 modulo p, such that $\mathrm{Ker}(\theta)$ the intersection of the normalizers of the Sylow p-subgroups and for any choice of a, b of the set of h elements, there is an element $g \in G$ such that $\theta(g)$ maps a to b.

4. Let M be a finitely generated module over the principal ideal domain D. Suppose that $d \in D - \{0\}$ is such that $dM = 0$. Prove that the D-module $\mathrm{Hom}_D(M,\, D/dD)$ of D-linear maps from M to D/dD is isomorphic to M as a D-module.

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5. Let V be an n-dimensional vector space over a field K, and let $T : V \to V$ be K-linear.

(a) Suppose that the matrix of T with respect to the basis $v_1, \ldots, v_n$ for V is upper triangular. For $i \le 1 \le n$, describe an ordered basis for $\bigwedge^i V$ in terms of $v_1, \ldots, v_n$ such that the matrix of $\bigwedge^i T : \bigwedge^i V \to \bigwedge^i V$ with respect to this ordered basis is upper triangular.

(b) Describe the diagonal entries of the matrix for $\bigwedge^i T$ in terms of the diagonal entries of the matrix for T.

(c) Let c_i denote the trace of the matrix $\bigwedge^i T : \bigwedge^i V \to \bigwedge^i V$, $1 \le i \le n$. Describe the relation of the coefficients of the characteristic polynomial $\det(xI - T)$ to the c_i.