UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra
9 May 2009: Morning Session, 9:00-12:00

1. Suppose that a group G acts on a set X. For a point $x \in X$, let G_x denote the stabilizer subgroup $\{g \in G \,|\, gx = x\}$.

a). If G acts transitively on X, show that G_x and G_y are isomorphic subgroups of G.

b). If G_x is conjugate to some subgroup H of G, show that $H = G_y$ for some $y \in X$.

c). Prove that if G acts transitively on X and G_x is normal, then $G_x = G_y$ for all x and y in X.

d). Show that if X is finite and G acts transitively, then the cardinality of X is equal to the cardinality of G/G_x, where $x \in X$ is any point.

2. Let λ be a non-zero element of the algebraically closed field k, and let R be the ring $k[t]/(t^2(t-\lambda)^3)$. Note that R is also a k-vector space in a natural way.

a). What is the dimension of R over k.

b). Find an explicit basis for the null space (kernel) of the linear transformation of R over k given by multiplication by $(t-\lambda)^3$.

c). Find the Jordan canonical form for the linear transformation of the vector space R given by multiplication by t.

3. Let $M_{n,m}(k)$ be the vector space of $n \times m$ matrices over a field k and consider the "matrix multiplication map"

$$B : M_{n,1}(k) \times M_{1,m}(k) \longrightarrow M_{n,m}(k)$$

$$\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} w_1 & \dots & w_m \end{pmatrix} \right) \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} w_1 & \dots & w_m \end{pmatrix}$$

a). Prove that the image of B consists precisely of the $n \times m$ matrices of rank at most one.

b). Explain why B is bilinear and use this show that $M_{n,1}(k) \otimes M_{1,m}(k) \cong M_{n,m}(k)$, as vector spaces over k.

c). Give (with justification) an explicit element of $k^3 \otimes k^3$ which can not be written as $v \otimes w$ for any $v, w, \in k^3$.

4. Compute the number of abelian groups of order 120, up to isomorphism.

5. If the Galois group of a finite Galois extension L/K is S_3, describe explicitly the lattice of intermediate fields. For each intermediate field F, specify what the Galois group of L/F is, and whether F is a normal extension of K.

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9 May 2009: Afternoon Session, 2:00-5:00

1. An ideal I in a commutative ring R is said to be radical if $f^n \in I$ implies $f \in I$, for any element $f \in R$ and any positive integer n.

a). State and prove a characterization of the radical ideals of $\mathbb{Z}$, in terms of their generators.

b). If I and J are radical ideals of $\mathbb{Z}$, prove that $I \cap J$ and $I + J$ are radical. Here, $I \cap J$ is the largest ideal contained in both I and J, and $I + J$ is the smallest ideal containing both I and J.

c). Prove or give a counterexample: If I and J are radical ideals in $\mathbb{Z}$, so is IJ. Here, IJ is the ideal generated by elements of the form xy where $x \in I$ and $y \in J$.

2. Let $p_1, \ldots, p_n$ be distinct prime integers, and let K be the extension of $\mathbb{Q}$ obtained by adjoining the square roots of these elements.

a). Describe the Galois group G of K over $\mathbb{Q}$ by giving an explicit set of generators.

b). Prove that G is abelian, and express it as a direct sum of subgroups of prime power order.

3. Let a be a real number. Find the rank and signature of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & a \end{pmatrix}$ as a function of a. For which values of a is it positive definite?

4. Give examples of the following:

a). A UFD R that is not a PID.

b). A module M over a PID that is neither free nor torsion.

c). A torsion module over a PID which has two submodules M and N satisfying $M \cap N = 0$ and whose annihilators satisfy $\operatorname{Ann} M \subset \operatorname{Ann} N$.

5. a). Let n be a positive integer, and let $\operatorname{Aut}(\mathbb{Z}/(2^n))$ be the automorphism group of the cyclic group $\mathbb{Z}/(2^n)$ of order 2^n. How many elements does $\operatorname{Aut}(\mathbb{Z}/(2^n))$ have?

b). Suppose that a finite group G has a cyclic 2-Sylow subgroup H. Show that the centralizer subgroup $Z_G(H)$ of H in G is equal to the normalizer subgroup $N_G(H)$ of H in G.