**UNIVERSITY OF MICHIGAN**
**DEPARTMENT OF MATHEMATICS**
**Qualifying Review Examination in Algebra**
*5 September 2009: Morning Session, 9:00-12:00*

1. Let $K$ be a finite normal separable extension of a field $F$, with Galois group $G$. For each case below, find the number of distinct intermediate fields $E$ such that $F \subset E \subset K$, with $E \neq K$ and $E \neq F$

- Case 1: $G$ is cyclic of order $d$ with $d > 1$.
- Case 2: $G$ has order 8 and $g^2 = \mathrm{id}$ for every element of $G$.
- Case 3: $G \cong D_7$, the symmetry group of a regular 7-gon.

2. How many elements of order seven are there in a simple group of order 168?

3. Find the Jordan canonical form of a $7 \times 7$ matrix $A$ whose characteristic polynomial is $(x-2)^4(x-5)^3$, whose minimal polynomial is $(x-2)^2(x-5)^2$, and such that the dimension of the kernel of the matrix $2I - A$ is three (here $I$ is the $7 \times 7$ identify matrix).

4. Count the number of prime ideals in the ring

$$\frac{\mathbf{Z}[x,y]}{(6,(x-2)^2,\, y^6)},$$

giving an explicit set of generators for each. Which of these contain the class of $x$?

5. Let $E, F$ and $G$ be finite dimensional real vector spaces, and suppose that $B : E \times F \to G$ is a bilinear form.

a) Is the image of $B$ necessarily a vector space? Prove or give a counterexample.

b) For $v \in F$, define $\phi_v : E \to G$ by $\phi_v(w) = B(w, v)$. Does there exist a bilinear form $B$ and nonzero vectors $v, w \in F$ such that $\phi_v$ and $\phi_w$ do not have the same rank? Prove or disprove.

**UNIVERSITY OF MICHIGAN**
**DEPARTMENT OF MATHEMATICS**
**Qualifying Review Examination in Algebra**
*5 September 2009: Afternoon Session, 2:00-5:00*

1. Let $G$ be a finite abelian group that contains 8 elements of order three, 18 elements of order nine, and no other elements besides the identify. Describe all possibilities for $G$, by giving explicit decompositions into cyclic groups, up to isomorphism.

2. The characteristic polynomial and the minimum polynomial of a linear transformation $T : V \to V$ of complex vector spaces are both equal to $(x-1)^2(x-b)^2$, where $b$ is a complex number.
a) Find the characteristic polynomial of the induced map $\wedge^2 T : \wedge^2 V \to \wedge^2 V$.
b) Find the rank of $\wedge^2 T$ as a function of $b$.

3. a) Show that the polynomial $x^6 + 3$ is irreducible over $\mathbb{Q}$.
b) Suppose that $\alpha$ is a root of $x^6 + 3$. Show that $\beta := (\alpha^3 + 1)/2$ is a primitive 6th root of unity.
c) Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension.
d) Determine the Galois group of this extension.

4. Let $R$ be a UFD, and let $f$ be any non-zero, non-unit irreducible element of $R$. Is $R/(f)$ also a UFD? Prove or give a counterexample. What happens if $R$ happens to also be a PID?

5. Let $p$ be a prime.
a) Suppose that $G$ is a finite group and $H$ is a normal subgroup. Let

$$Z_G(H) = \{g \in G \mid \forall h \in H\, gh = hg\}$$

be the centralizer of $H$ in $G$. Show that $Z_G(H)$ is a normal subgroup of $G$ and that $G/Z_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$, the group of automorphisms of $H$.
b) Show that every group of order $p^4$ has an abelian normal subgroup of order $p^2$.
c) Show that every group of order $p^4$ has an abelian normal subgroup of order $p^3$.