UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
7 September 2009: Morning Session, 9:00-12:00

Problem 1: Compute

$$\int_0^{2\pi} \frac{d\theta}{(3+e^{-i\theta})^2}$$

by converting this integral to a contour integral over the unit circle and applying the Residue Theorem.

Problem 2: Let f be a real Lebesgue measurable function on the interval $[0,1]$ such that $\|f\|_\infty < \infty$. Show that for any $\varepsilon, \delta > 0$, there is a continuous function g on $[0,1]$ such that $m\{x \in [0,1] : |f(x) - g(x)| > \varepsilon\} < \delta$.

Problem 3: Exhibit an entire function $f : \mathbb{C} \to \mathbb{C}$ satisfying $|f(z)| > e^{|z|}$ for $|z| > 10$ or explain why no such function exists.

Problem 4: Suppose that $f(x)$, $x > 0$, is a real valued Lebesgue measurable square integrable function.

(a) Prove that for any $\alpha > 0$, the inequality $2|f(z)||f(y)| \le \alpha f(z)^2 + f(y)^2/\alpha$ holds for all $z, y, \alpha > 0$.

(b) Express the double integral

$$\int_0^\infty \int_0^\infty \frac{|f(z)||f(y)|}{y+z}\, dz\, dy$$

as an integral over the region $\{0 < z < y < \infty\}$.

(c) Show using your work from (a) and (b) that $|f(z)||f(y)|/(y+z)$, $y, z > 0$, is integrable and

$$\int_0^\infty \int_0^\infty \frac{|f(z)||f(y)|}{y+z}\, dz\, dy \;\le\; 4 \int_0^\infty f(x)^2\, dx.$$

Hint: Use the inequality in (a) with $\alpha = (z/y)^{1/2}$.

Problem 5: Find a conformal map from $D_1 = \{x + iy \;:\; 0 < y < 1\}$ onto $D_2 = \{x + iy \;:\; y > |x|\}$.

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7 September 2009: Afternoon Session, 2:00-5:00

Problem 6: Suppose that $f_n(x),\ n = 1, 2, ...,$ is a sequence of positive Lebesgue measurable functions on the interval $[0,1]$ with the property that $\|f_n\|_\infty \le 5,\ n = 1, 2, ...$ Assume in addition that there is a Lebesgue measurable function $f(x),\ x \in [0,1]$, such that $\lim_{n\to\infty} m\{x \in [0,1] : f_n(x) > \alpha\} = m\{x \in [0,1] : f(x) > \alpha\}$ for any $\alpha > 0$. Prove that $f(\cdot)$ is Lebesgue integrable on $[0,1]$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \int_0^1 f(x)\ dx.$$

Problem 7: Find explicit radii $r_2 > r_1 > 0$ so that the polynomial $p(z) = z^4 + 100z^3 + 2$ has exactly three roots in the annulus $r_1 < |z| < r_2$. Are the three roots distinct?

Problem 8: Consider the sequence $f_n(x) = (\sin(\pi n x))^n,\ n = 1, 2, ...,$ on the interval $[0,1]$. Prove that for any $\delta > 0$ there is a set $E \subset [0,1]$ with $m(E) > 1-\delta$, and a subsequence $f_{n_k}(x),\ k = 1, 2, 3...,$ such that $\lim_{k\to\infty} f_{n_k}(x) = 0$ for $x \in E$.

Problem 9: Let $f(x+iy)$ be an analytic function on the half-plane $x > 0$ satisfying the estimate $|f(x+iy)| \le x^{-2}$. Find an explicit upper bound for $|f'(1)|$.

Problem 10: Let f be a real Lebesgue measurable function on the interval $[0,1]$ such that $\|f\|_\infty < \infty$. For $\alpha \in \mathbb{R}$ define a function $g(\alpha)$ by

$$g(\alpha) = \log\left[\int_0^1 \exp[\alpha f(x)]\ dx\right].$$

(a) Prove that the function $g(\cdot)$ is twice continuously differentiable and that $g''(\alpha) \ge 0$ for all $\alpha \in \mathbb{R}$, i.e. the function $g(\cdot)$ is convex.

(b) Prove that if f is a *non-constant* function,
i.e. $m\{x \in [0,1] : |f(x) - c| \ne 0\} > 0$ for all constants $c \in \mathbb{R}$, then $g''(\alpha) > 0,\ \alpha \in \mathbb{R}$.