UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology
3 January 2009: Morning Session, 9:00-12:00

Problem 1. Let X be a compact metric space, and let Y be a Hausdorff space. Suppose that there is a continuous mapping f of X onto Y. Show that Y has a countable basis.

Problem 2. Let X be a compact Hausdorff space with the property that each of its points is a G_δ set, i.e. is the intersection of countably many open subsets. Prove that X is "first countable", i.e. has a countable basis at each of its points.

Problem 3. For all $a \in \mathbb{R}$ let Z_a be the subset of those $(x, y, z) \in \mathbb{R}^3$ with $(x^2 + y^2)z = a$.

- Determine for which a is Z_a a topological manifold.
- Determine for which a is Z_a connected.
- Determine for which a is Z_a simply connected.
- Determine for which a is Z_a contractible.

Problem 4. Identify the 2-dimensional sphere $\mathbb{S}^2$ with the subset of $\mathbb{R}^3$ consisting of vectors of length 1 and for $x, y \in \mathbb{S}^2$ let $d(x, y)$ be their distance in $\mathbb{R}^3$. Prove that if X is any topological space and $f, g : X \to \mathbb{S}^2$ are such that $d(f(x), g(x)) < 2$ for all $x \in X$, then f and g are homotopic to each other.

Problem 5. Let M be a simply connected manifold. Prove that every continuous map $f : M \to \mathbb{S}^1 \times \mathbb{S}^1$ is homotopic to a constant map.

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Problem 6. Let X be a compact Hausdorff space, and let $\cdots \subset X_n \subset X_{n-1} \subset \cdots \subset X_2 \subset X_1$ be a nested sequence of closed, nonempty connected subsets of X. Prove that $\cap_{i=1}^{\infty} X_i$ is nonempty and connected.

Problem 7. Let N and M be compact manifolds without boundary and of the same dimension. Prove that any immersion $f : N \to M$ is a covering map.

Problem 8. Let M be a connected manifold with $\pi_1(M)$ finite with odd order. Prove that there is no degree 2 cover $\pi : N \to M$ with N connected. Deduce that N is orientable.

Problem 9. For $g = 0, 1, 2, \ldots$ let Σ_g be the compact orientable surface of genus g with empty boundary. For which g is there a non-trivial covering map $f : \Sigma_g \to \Sigma_g$?

Problem 10. Let X and Y be two compact manifolds, $A \subset X$ and $B \subset Y$ two closed connected submanifolds, and $f : A \to B$ a homeomorphism. Denote by $Z = X \cup_f Y$ the space obtained by identifying $x \in A$ with $f(x) \in B$. Compute the Euler-characteristic $\chi(Z)$ of Z in terms of $\chi(X)$, $\chi(Y)$ and $\chi(A)$.