

UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology
8 May 2009: Morning Session, 9:00-12:00

Problem 1. Let $C^0(\mathbb{R}, \mathbb{R})$ denote the set of continuous functions

$$f : \mathbb{R} \rightarrow \mathbb{R}.$$

For a compact set $K \subset \mathbb{R}$, $\epsilon > 0$ and $f \in C^0(\mathbb{R}, \mathbb{R})$, define

$$B_\epsilon(f, K) = \{g \in C^0(\mathbb{R}, \mathbb{R}) \mid \sup_{x \in K} \{|f(x) - g(x)|\} < \epsilon\}.$$

The set of all subsets of $C^0(\mathbb{R}, \mathbb{R})$ of the form $B_\epsilon(f, C)$ is a basis for the *topology of compact convergence* on $C^0(\mathbb{R}, \mathbb{R})$. From now on we consider $C^0(\mathbb{R}, \mathbb{R})$ to be endowed with this topology.

- (a) Show that $C^0(\mathbb{R}, \mathbb{R})$ is Hausdorff, second countable, connected and simply connected.
- (b) Which of the following two sets are compact in the topology of compact convergence?
 - $\{f_n(x) = x + \sin(nx), n \in \mathbb{N}\}$
 - The set of all polynomials of degree at most 4 all whose coefficients have absolute value less than 1.

Problem 2. Let X be the space obtained from $\mathbb{S}^1 \times \mathbb{R}$ by removing the interior of k disjoint 2-disks.

- (a) Compute the fundamental group $\pi_1(X)$.
- (b) What would be your answer to part (a) if $\mathbb{S}^1 \times \mathbb{R}$ is replaced by $\mathbb{S}^2 \times \mathbb{R}$ and 2-disks are replaced by 3-balls?
- (c) Let Y be the union of two copies of the real projective plane $\mathbb{R}P^2$ having exactly one point y in common. Compute $\pi_1(Y, y)$.

Problem 3. Let $\mathbb{S}^1 = \{(x, y, 0, 0) \in \mathbb{R}^4 \mid x^2 + y^2 = 1\}$ be the unit circle and consider $M = \mathbb{R}^4 \setminus \mathbb{S}^1$. Compute the fundamental group $\pi_1(M)$ and the homology groups $H_*(M)$ of M .

Problem 4.

- (a) Give an example of a local homeomorphism $\mathbb{R}^2 \rightarrow \mathbb{S}^2$ which is surjective but not a covering.
- (b) Prove that there is no local homeomorphism $\mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

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Problem 5.

- (a) For $n \geq 1$, let $\mathbb{R}P^n$ be the real projective space. Compute $\pi_1(\mathbb{R}P^n)$.
- (b) Let $\tau : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the homeomorphism $\tau(x) = -x$, let $\langle \tau \rangle$ be the cyclic group generated by τ and endow $\mathbb{R}^3/\langle \tau \rangle$ with the quotient topology. Prove that $\mathbb{R}^3/\langle \tau \rangle$ is not a manifold.

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Problem 6. Endow $\{0, 1\}$ with the discrete topology and $X = \{0, 1\}^{\mathbb{N}}$ with the product topology.

- (a) Prove that X is compact and totally disconnected.
- (b) Construct a continuous surjective map $\pi : X \rightarrow [0, 1]$ such that for every $t \in [0, 1]$ the set $\pi^{-1}(t)$ consists of at most 2 points.

Problem 7. Let S_1 and S_2 be two closed orientable surfaces and $\gamma_1 \subset S_1$ and $\gamma_2 \subset S_2$ be two simple closed curves. Let X be the space obtained by gluing S_1 and S_2 along γ_1 and γ_2 . Is X a manifold? If not, is X homotopy equivalent to a manifold?

Problem 8. Compute $H_*(\mathbb{R}P^2 \# \mathbb{R}P^2)$ where $\mathbb{R}P^2$ is the real projective plane and $\#$ denotes connected sum.

Problem 9. Let X be a path-connected space with base point x_0 , and consider the function

$$\phi : \pi_1(X, x_0) \rightarrow H_1(X, \mathbb{Z})$$

which takes a homotopy class $[\alpha]$ to the singular homology class $\alpha_*([\mathbb{S}^1])$ where $[\mathbb{S}^1] \in H_1(\mathbb{S}^1, \mathbb{Z})$ is a fixed generator. Show that

- (a) ϕ is a homomorphism, and
- (b) ϕ is surjective.

Problem 10. Let SO_n be the set of orthogonal n -by- n matrices with real coefficients and determinant 1. Consider SO_n as a subset of \mathbb{R}^{n^2} .

- (a) Prove that SO_n is a manifold.
- (b) Show that SO_n admits a nowhere vanishing vectorfield.
- (c) Compute $\chi(\text{SO}_n)$ (Hint: use (b)).