UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Topology

12 September 2009: Morning Session, 9:00-12:00

1. Prove that if X is a (non-empty) countable compact Hausdorff space, then X is not connected. (You may use the fact that an intersection of countably many dense open sets in a compact Hausdorff space is dense.)

2. Let P be a polygon with an even number of sides. Suppose that the sides are identified in pairs in any way whatsoever. Prove that the quotient space is a manifold.

3. Prove that if M is a non-empty compact smooth manifold with boundary, then there is no smooth retraction from M to its boundary ∂M. (You may use Sard's theorem.)

4. Let X be a path-connected topological space. For $n > 1$ an integer, denote by S_n the symmetric group on n-letters. State and prove a bijective correspondence between degree n covering spaces of X and group homomorphisms $\pi_1(X) \to S_n$. (Note that finding an accurate statement is part of the problem.)

5. For integers k, n with $1 \leq k \leq n$, let

$$S^n = \{(x_1, ..., x_{n+1}) | x_1^2 + ... + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

and let

$$D_k = \{(x_1, ..., x_{n+1}) | x_1^2 + ... + x_k^2 \leq 1, \ x_{k+1} = ... = x_{n+1} = 0\}.$$

Calculate the homology of $X_{k,n} = S^n \cup D_k$.

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12 September 2009: Afternoon Session, 2:00-5:00

1. Prove that the one point compactification $X \cup \{\infty\}$ is Hausdorff if and only if X is locally compact and Hausdorff.

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. The point $(x, y) \in \mathbb{R}^2$ is the stereographic projection of the point $(\xi, \eta, \zeta) \in S^2$ if and only if the three points $(0, 0, 1), (x, y, 0)$, and (ξ, η, ζ) are collinear; this defines a map $\sigma : \mathbb{R}^2 \to S^2, \sigma(x, y) = (\xi, \eta, \zeta)$. Show that σ maps $\mathbb{R}^2$ diffeomorphically onto the complement of a point in S^2.

3. By definition, a topological group is a set G with both a topology and a group structure, such that the map $G \to G$ sending x to x^{-1} and the map $G \times G \to G$ sending (x, y) to xy are both continuous. Let $1 \in G$ denote the identity of this topological group G. Show that $\pi_1(G, 1)$ is abelian.

4. Show that the map $\phi : S^1 \times S^1 \to \mathbb{R}^3$ defined by

$$\phi(u, v) = \begin{pmatrix} cos\alpha & sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 + cos\beta \\ 0 \\ sin\beta \end{pmatrix}$$

   for $u = (cos\alpha, sin\alpha)$ and $v = (cos\beta, sin\beta)$ is an embedding.

5. Let X be a finite simplicial complex of dimension 1. Prove that either $\pi_1 X \cong \mathbb{Z}$, or every continuous map $f : X \to X$ homotopic to the identity has a fixed point.