UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

6 January 2007: Morning Session, 9:00-12:00

(AM1) Let G be a group and suppose that G acts on the set Ω. Let $k \geq 1$ be an integer. We say G is *k-fold transitive* if, given any two ordered sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of distinct elements of Ω, there is an element $g \in G$ so that $g(x_i) = y_i$ for $i = 1, \ldots, k$. Suppose that G, Ω, and k are as above, with a k-fold transitive action of G on Ω.

(a) Suppose that Γ is a subset of Ω (not ordered) with k elements. Let H be the subgroup of G that maps Γ to Γ. Prove that the natural action of H on Γ gives a homomorphism of H onto Sym_k, the symmetric group on k elements.

(b) Prove that G is not solvable or that $k \leq 4$.

(AM2) Let p be a prime number, $q = p^n$ a power of p, and let $\mathbf{F}_q$ be the finite field with q elements.

(a) Simplify the polynomial $f(x) := \prod_{c \in \mathbf{F}_q} (x - c)$. Justify your answer.

(b) Let E be a finite degree extension field of $\mathbf{F}_q$ and $r \in E$. Let $g(x)$ be the minimal polynomial of r over the prime field $\mathbf{F}_p$. Give a simple criterion involving only the degree d of $g(x)$ and q for r to be in $\mathbf{F}_q$.

(AM3) Let $A_1, \ldots, A_m$ be mutually commuting linear transformations on a vector space V of finite dimension $n \geq 1$ over a field K such that $A_1 \cdots A_m = 0$. Show that there are t distinct indices $i_1, \ldots, i_t$ with $t \leq n$ such that $A_{i_1} \cdots A_{i_t} = 0$.

(AM4) Let G be a group of order 8 containing 6 elements of order 4 and one element of order 2.

(a) How many subgroups does G have of each possible order? For each order, how many of these are normal?

(b) Let L be a finite degree Galois extension field of the field K with Galois group G of order 8 as above. For each $d \leq 8$, determine how many intermediate fields F have degree d over K, and determine as well how many of these fields are normal extensions of K.

(AM5) Let S be a $p \times p$ matrix and T a $q \times q$ matrix, both over the complex numbers. There is a well-defined linear transformation $S \otimes T$ on $\mathbf{C}^p \otimes \mathbf{C}^q \cong \mathbf{C}^{pq}$ which satisfies $(S \otimes T)(u \otimes v) = S(u) \otimes T(v)$ for $u \in \mathbf{C}^p$ and $v \in \mathbf{C}^q$. Prove that $\mathrm{Trace}(S \otimes T) = \mathrm{Trace}(S)\mathrm{Trace}(T)$ and that $\det(S \otimes T) = \big(\det(S)\big)^q \big(\det(T)\big)^p$.

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DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

6 May 2004: Afternoon Session, 2:00-5:00

(PM1) Describe, using Jordan canonical forms, the possible similarity classes for a 4×4 matrix M over the complex number $\mathbf{C}$ given that M satisfies $M^4 = M^2$.

(PM2) (a) Define the term *nilpotent group* using one of the notions of central series.

(b) Prove that a group which is the product of two normal, abelian subgroups (which may have non-trivial intersection) has a central series of length at most 2.

(PM3) Let R be the PID $\mathbf{F}_2[x]$, where $\mathbf{F}_2$ is the field with 2 elements and x is an indeterminate, and let $D = R^4$, the free module of length 4 row vectors over R. Define E to be the row space of the matrix

$$A := \begin{pmatrix} x & 0 & 0 & 0 \\ x^4 + x^3 & x+1 & x^3 + x & x^2 \\ x^3 + x^2 + x + 1 & 0 & x^2 + 1 & x^2 + x \\ 0 & 0 & 0 & x \end{pmatrix}$$

Recall that the annihilator of a module M over a ring R is the ideal $\{r \in R : rm = 0 \text{ for all } m \in M\}$.

(a) Describe a sequence of cyclic modules $C_1, \ldots, C_h$ so that D/E is isomorphic to the direct sum of the C_i and the annihilators of the C_i are linearly ordered by inclusion.

(b) Describe a sequence $M_1, \ldots, M_k$ of primary cyclic modules (these are cyclic modules whose annihilator is a power of a prime ideal) so that D/E is isomorphic to the direct sum of the M_j.

(PM4) Let $\mathbf{Q}$ be the rational numbers. Let $f(x)$ be a polynomial of positive degree over $\mathbf{Q}$, and let G be the Galois group of K, the splitting field of $f(x)$ over $\mathbf{Q}$.

(a) Let L be any field intermediate between $\mathbf{Q}$ and K and let H be the Galois group of the splitting field of $f(x)$ over L. What is the relationship between H and G? (This should be a simple statement.) Can all choices of H that have this relationship to G occur? Why?

(b) Let p be an odd prime integer, and let $f(x) = x^p - 2$. Find the Galois group G of the splitting field of $f(x)$ over $\mathbf{Q}$. Show that it is the product of two cyclic subgroups with trivial intersection, exactly one of which is normal.

(PM5). (a) State the Sylvester theorem for real symmetric bilinear forms, which assigns a "signature" to such a form.

(b) Let B be a non-degenerate real symmetric bilinear form in n variables. Prove that there is a subspace $U \subseteq \mathbf{R}^n$ of even dimension $2d$ with basis $u_1, \ldots, u_{2d}$ such that the matrix $\big(B(u_i, u_j)\big)$ is a block-diagonal sum of d copies of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and such that the annihilator $U^\perp$ of U in $\mathbf{R}^n$ does not contain a nonzero vector v such that $B(v, v) = 0$. Express d in terms of n and the signature of B.