UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

3 September 2007: Morning Session, 9:00-12:00

Answer all questions with full explanations. $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are the integers, rational numbers, real numbers, and complex numbers, respectively. $\mathbb{Z}_n$ denotes the ring of integers modulo n.

1. A module M over the ring of integers $\mathbb{Z}$ is the quotient of the free module F on four generators e_1, e_2, e_3, e_4 by the submodule G generated by the elements:

$$e_1 + 2e_2 + 3e_3 + 5e_4$$
$$2e_1 + e_2 + 6e_3 + 4e_4$$
$$e_1 - e_2 + 3e_3 - e_4$$

What is F/G as a direct sum of cyclic modules? What is the torsion-free rank of F/G? What is G as a direct sum of cyclic modules?

2. Let K be a field and let A, B, and C denote $n \times n$ matrices over K, where $n \geq 2$. Let $\mathbf{1}$ denote the $n \times n$ identity matrix.

(a) Is it possible that $AB = BA = 0$ and that $A + cB$ is invertible for every element $c \neq 0$ in K? Give an example or prove that this is not possible.

(b) Suppose that the characteristic of K is different from 2. Prove that if $C^2 = 0$, then $\mathbf{1} + C$ is the square of an $n \times n$ matrix.

3. Let $p > 0$ be an odd prime integer. The group D of symmetries of a regular p-sided polygon consists precisely of p elements of order 2, $p-1$ elements of order p, and the identity element: you may assume this. Suppose that F is a finite algebraic Galois extension of the rational numbers $\mathbb{Q}$, and that its Galois group is D.

(a) What is $[F : \mathbb{Q}]$?

(b) How many fields are there that are strictly intermediate between $\mathbb{Q}$ and F? What are the degrees of these fields over $\mathbb{Q}$?

(c) How many of the intermediate fields in part (b) are normal field extensions of $\mathbb{Q}$, and what are their degrees over $\mathbb{Q}$?

(d) Are the elements of F expressible by radicals over $\mathbb{Q}$? Explain.

4. Let K be a field and V a finite-dimensional vector space over K of dimension n.

(a) Prove that $v_1, \ldots, v_h \in V$ are linearly dependent if and only if $v_1 \wedge \cdots \wedge v_h = 0$.

(b) If $v_1, \ldots, v_n$ is a basis for V, describe a basis for $\bigwedge^k V$, $1 \leq k \leq n$, in terms of these. What is the cardinality of this basis? What is $\bigwedge^k V$ if $k > n$?

(c) Let u, $v \in V$ be linearly independent vectors, and let u', $v' \in V$ be linearly independent vectors. Prove that $u \wedge v$ and $u' \wedge v'$ span the same line in $\bigwedge^2(V)$ if and only if the plane spanned by u and v is the same as the plane spanned by u' and v'.

5. Describe, up to isomorphism, all groups G with identity e generated by two elements x and y such that $x^3 = e = y^7$, $x \neq e$, $y \neq e$, and $xy = y^k x$ for some integer k with $1 \leq k \leq 6$. In particular, which values of k are possible?

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DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Algebra

3 September 2007: Afternoon Session, 2:00-5:00

Answer all questions with full explanations. $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are the integers, rational numbers, real numbers, and complex numbers, respectively. $\mathbb{Z}_n$ denotes the ring of integers modulo n.

1. Let S_7 denote the group of all permutations of the set $\{1, 2, 3, 4, 5, 6, 7\}$. Let $g = (1\ 2\ 3)(4\ 5\ 6\ 7) \in S_7$, the product of a 3-cycle and a 4-cycle that are disjoint. Let G denote the subgroup of S_7 generated by g.

(a) What is the order of G?

(b) How many conjugates does g have in S_7?

(c) What is the order of the centralizer of g in S_7?

(d) How many conjugates does G have in S_7?

2. Describe the maximal ideals m of the polynomial ring $\mathbb{Z}[x]$ in one variable over the integers that contain the integer 30 and the polynomial $x^2 + 1$. Give explicitly two generators for each such maximal ideal m. How many such maximal ideals are there?

3. For every real number $a \in \mathbb{R}$, let $B_a : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ denote the symmetric bilinear form with matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 9 \\ 3 & 7 & 11 & 13 \\ 4 & 9 & 13 & a \end{pmatrix}$$

Determine the rank and signature of B_a for every choice of a. Give the definition of the notion of signature that you are using. For which values of a is B_a nondegenerate? For which values of a is B_a positive definite?

4. Let A and B denote $n \times n$ matrices over $\mathbb{C}$, and suppose that there are integers $r > 1$, $s > 1$ such that $A^r = B$ and $B^s = A$. Prove that A and B must be diagonalizable, or give a counterexample.

5. Let $K = GF(q)$ denote the finite field with $q = p^k$ elements, where p is a prime integer ≥ 2.

(a) Suppose that R is a finite ring containing K and that the identity for K is an identity for R. What restriction does this place on the cardinality of R? Show that if r is an integer that satisfies this restriction, then there is a field with r elements that contains K.

(b) $GF(q)$ can be obtained as the splitting field of a polynomial over $\mathbb{Z}_p$. Describe such a polynomial explicitly.

(c) Assume that we have $K = GF(q) \hookrightarrow GF(r) = L$. Say as much as you can about the Galois group of L over K. Specifically, what are the automorphisms of L over K?

(d) Prove that the number of subfields of $GF(p^n)$ is the same as the number of positive integer divisors of n.