UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
4 January 2005: Morning Session, 9:00-12:00

1. Let $E \subset \mathbb{R}$ be measurable with $|E| < \infty$. Show that $|E \setminus [-R, R]| \to 0$ as $R \to \infty$.

2. Let $f_n(x)$, $n = 1, 2, \ldots$ and $f(x)$ be measurable functions in a measure space (X, μ) such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every x. Suppose that there are positive measurable functions $g_n(x)$, $n = 1, 2, \ldots$ and $g(x)$ such that $\lim_{n\to\infty} g_n(x) = g(x)$ for almost every x, $|f_n(x)| \le g_n(x)$ for all $n = 1, 2, \ldots$ and
$$\lim_{n\to\infty} \int_X g_n(x) d\mu(x) = \int_X g(x) d\mu(x).$$
Prove that
$$\lim_{n\to\infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x).$$

3. Let $f : U \to \mathbb{C}$ be analytic, $f(x + iy) = u + iv$. Show that u, v, uv are harmonic. How about u^2? Recall that $g(x + iy)$ is harmonic if $g_{xx} + g_{yy} = 0$.

4. Let
$$f(z) = \begin{cases} \frac{\sin z}{z \cos z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$
   (a) Show that $f(z)$ has the Taylor expansion of the form
   $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
   in a neighborhood of $z = 0$.
   (b) What is the radius of convergence of the series in question (a).
   (c) Show that $a_{2j+1} = 0$ for $j = 0, 1, 2, \ldots$. Also compute the coefficients a_0, a_2 and a_4.

5. Prove or give a counterexample to the following statements.
   (a) If $f \in L^p(\mathbb{R}, dx)$ for all $1 \le p < \infty$, then $f \in L^\infty(\mathbb{R}, dx)$.
   (b) If f is absolutely continuous in $\mathbb{R}$ and $f' \in L^1(\mathbb{R}, dx)$, then $f(x) \to 0$ as $x \to \infty$.

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6. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, show that there exists g for which $g = f$ a.e. and g is Borel measurable.

7. (a) Suppose that $f(z)$ is analytic for $1 \le |z| \le 3$. Assume that $|f(z)| \le 1$ for $|z| = 1$ and $|f(z)| \le 9$ for $|z| = 3$. Prove that $|f(2i)| \le 4$.

   (b) Prove that there is no non-constant analytic function on the Riemann sphere.

8. Show that for a continuous function $f(x)$, $0 \le x \le 1$,

$$\lim_{n\to\infty} \sqrt{n} \int_0^1 \frac{f(x)}{1+nx^2} dx = \frac{\pi}{2} f(0).$$

9. Suppose that $f(z)$ is analytic in a neighborhood of the interval $[-1, 1]$. Define

$$F(z) = \int_{-1}^{1} \frac{f(s)}{s-z} ds, \qquad z \in \mathbb{C} \setminus [-1, 1].$$

   (a) Show that $F(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$.

   (b) Show that for $-1 < x < 1$, the limits

$$F_+(x) := \lim_{\epsilon \downarrow 0} F(x + i\epsilon), \qquad F_-(x) := \lim_{\epsilon \downarrow 0} F(x - i\epsilon)$$

   exist and satisfy

$$F_+(x) - F_-(x) = 2\pi i f(x).$$

10. Let $\mathcal{F}$ consist of all holomorphic functions $f : \Delta \to \Delta$, Δ the unit disc. Let $L := \sup_{f\in\mathcal{F}} |f''(0)|$. Does there exist an $f \in \mathcal{F}$ for which $f''(0) = L$? Justify your answer.