UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
4 January 2006: Morning Session, 9:00-12:00

1. Let $t \in \mathbb{R}$. Evaluate the integal

$$I(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2} dx.$$

2. (a) Let $I \subset [0, 2\pi]$ be an interval. Prove that

$$\lim_{n \to \infty} \int_I \sin(nx) dx = 0.$$

   (b) Let $A \subset [0, 2\pi]$ be a measurable set. Prove that

$$\lim_{n \to \infty} \int_A \sin(nx) dx = 0.$$

3. Let $f(z)$ be analytic in $|z| \leq 1$. Set $A = \sup_{|z|=1} Re\big(f(z)\big)$.

   (a) Let $g(z) = \frac{f(z)}{2A - f(z)}$. Prove that $|g(z)| \leq 1$ for $|z| \leq 1$.

   (b) Suppose furthermore that $f(0) = 0$. Set $M(r) = \sup_{|z|=r} |f(z)|$, $0 < r < 1$. Prove that

$$M(r) \leq \frac{2Ar}{1-r}.$$

4. Let $f(z)$ be an entire function satisfying that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta \leq r^{10/3}$$

   for all $r > 0$. Prove that f is the zero function.

5. Let the sequence $\{x_n\}_{n=0}^{\infty}$ be defined inductively by $x_0 = -2$ and

$$x_{n+1} = \frac{x_n}{1 + \frac{1}{x_n}}, \qquad n \geq 0.$$

   (a) Prove that $x_n \to -\infty$ as $n \to \infty$.

   (b) Prove that

$$\lim_{n \to \infty} \frac{x_n}{n} = -1.$$

   (Hint: It might be useful to note that $\frac{1}{1+r} = 1 - r + O(r^2)$ as $r \to 0$.)

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6. Let $f(z) = z(1+z)$. Let $V = \{z \in \mathbb{C} : 0 < |z| < \frac{1}{2},\ \frac{7}{8}\pi < arg(z) < \frac{9}{8}\pi\}$.

   (a) Show that if $z \in V$, then $|f(z)| \le |z|$ and $f(z) \in V$.

   (b) Prove that if $z \in V$, then $\lim_{n\to\infty} f^n(z) = 0$ where $f^n(z) = f \circ f \circ \cdots \circ f(z)$ is the n-fold composition of f.

7. Let $f : [0,1] \to \mathbb{R}$ be a function with the property that there exists a constant $M > 0$ and $\alpha > 1$ such that $|f(x) - f(y)| \le M|x-y|^\alpha$ for all $x, y \in [0,1]$. Show that f is a constant function.

8. Find a function $f(z)$ in the complex plane satisfying the following properties:

   (i) $f(z)$ is analytic in the upper half plane $Im(z) > 0$, and is continuous up to the real axis except at the origin $z = 0$,

   (ii) $f(x)$ is real for $x \in \mathbb{R} \setminus \{0\}$,

   (iii) there is a constant $C > 0$ such that $|f(z)| \le \frac{C}{|z|^3}$ for all $Im(z) > 0$.

   (iv) $f(i) = 4i$.

   Also determine if this function is unique.

9. Let $\varphi \in L^1(\mathbb{R})$ satisfy $\varphi(x) \ge 0$ and $\int_{\mathbb{R}} \varphi(x)dx = 1$. Set $\varphi_t(x) = \frac{1}{t}\varphi\left(\frac{x}{t}\right)$ for $t > 0$. The convolution of two functions f, g on $\mathbb{R}$ is defined as $(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ where it is defined.

   (a) For $f \in L^1(\mathbb{R})$, prove that $f * \varphi_t \to f$ in L^1 as $t \downarrow 0$.

   (b) Let $1 < p < \infty$. For $f \in L^p(\mathbb{R})$, prove that $f * \varphi_t \to f$ in L^p as $t \downarrow 0$.

10. Determine if the following statements are true or false. Give your reason.

   (a) Let $E \subset \mathbb{R}$ be measurable and let $f : E \to \mathbb{R}$ be continuous. If $\lambda(E) < \infty$, then $\lambda(f(E)) < \infty$.

   (b) If u is harmonic in $\mathbb{C}$, then there exists an entire function f such that $Re(f(z)) = u(z)$.

   (c) There exists a non-constant, non-negative harmonic function on $\mathbb{C}$.

   (d) Let D be a domain in $\mathbb{C}$. Let f be analytic in D and let γ be a closed curve in D. Then $\int_\gamma f(z)dz = 0$.

   (e) If $\{K_j\}_{j=1}^\infty$ is a sequence of non-empty compact sets in $\mathbb{R}$ such that $K_{j+1} \subset K_j$ for each j, then $\cap_{j=1}^\infty K_j \ne \emptyset$.