

UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
2 January 2007: Morning Session, 9:00-12:00

1. Show that if $f \in L^3([-1, 1])$, then the integral

$$\int_{-1}^1 \frac{|f(x)|}{\sqrt{|x|}} dx$$

is finite.

2. Use the method of residues to calculate the integral

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx.$$

Provide details to justify each step in your computation.

3. Let $\{f_n\}$ be a sequence of measurable functions on a measure space (X, \mathcal{M}, μ) . Suppose that the infinite series

$$\sum_{n=1}^{\infty} \mu \{x \in X : |f_n(x)| \geq \varepsilon\}$$

converges for each $\varepsilon > 0$. Prove that $f_n(x) \rightarrow 0$ a.e.

4. Construct a conformal mapping f of the domain

$$\Omega = \{z \in \mathbb{C} : |z| < 1 \text{ and } |z - \frac{1}{2}| > \frac{1}{2}\}$$

onto the unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$, with $f(-\frac{1}{2}) = 0$. Draw figures to illustrate each step of the construction.

5. Let f and g be nonnegative functions in $L^1(\mathbb{R})$. Suppose that each function is positive on some set of positive measure. (However, there need not be a single set of positive measure where both functions are positive.) Prove that the convolution

$$h(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt$$

is positive on some set of positive measure.

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6. (a) Prove that a power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 1/\alpha$, where

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

In other words, show that the series converges for $|x| < R$ and diverges for $|x| > R$. Here it is understood that $R = 0$ if $\alpha = \infty$, and that $R = \infty$ if $\alpha = 0$.

(b) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} (\cos n)^n x^n,$$

and explain your reasoning.

7. Let $\Omega \subset \mathbb{C}$ be a bounded domain, and let \mathcal{F} be a family of functions analytic and uniformly bounded in Ω . Show that the functions $f \in \mathcal{F}$ satisfy a uniform Lipschitz condition $|f(z_1) - f(z_2)| \leq C|z_1 - z_2|$ in each subdomain $D \subset \overline{D} \subset \Omega$, where the constant C depends only on D and not on f .

8. Prove that if f is analytic in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, then

$$f\left(\frac{1}{n}\right) \neq \frac{1}{n+2} \quad \text{for some integer } n = 2, 3, \dots$$

9. Let $f(z)$ be an analytic function defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ so that $f(0) = 0$ and $-1 < \operatorname{Re}\{f(z)\} < 1$ for all $z \in \mathbb{D}$. Prove that

$$|\operatorname{Im}\{f(z)\}| \leq \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}.$$

10. *Prove or disprove:* If $\{r_n\}$ is an enumeration of the rational numbers in the interval $(0, 1)$, there exists a number x in $(0, 1)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/2}} < \infty.$$