UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Analysis

7 May 2005: Morning Session, 9:00-12:00

Name: ______________________________

1. Let $E \subset \mathbb{R}$ be a measurable subset. Find a necessary and sufficient condition on E that $L^2(E) \subset L^1(E)$.

2. Decide if the functions below are harmonic and if they are find their harmonic conjugates.

   (a)
   $$u(x, y) = xy - x^3 + 3xy^2$$

   (b)
   $$u(x, y) = x^4 + 6x^2y^2 - y^4$$

3. Let $f \in L^1(\mathbb{R}, dx)$. Prove that to each $\epsilon > 0$ there is $\delta > 0$ such that $\int_E |f(x)|ds < \epsilon$ for any set $E \subset \mathbb{R}$ satisfying $|E| < \delta$.

4. Consider the function
   $$h(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

   (a) Show that $h(z)$ is analytic when $z \neq n, n \in \mathbb{Z}$.

   (b) Show that h has removable singularities at each $z \in \mathbb{Z}$.

   (c) Noting that $h(z+1) = h(z)$, show that $h(z)$ is bounded in the complex plane.

   (d) Show that $h(it) \to 0$ as $t \to \infty, t > 0$.

   (e) Show that hence $h(z) \equiv 0$, and that this implies that
   $$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

5. Let μ and λ be positive measures on a σ-algebra $\mathcal{A}$. Prove that the two following statements are equivalent:

   (a) λ is absolutely continuous with respect to μ (i.e. $\lambda(A) = 0$ for every $A \in \mathcal{A}$ such that $\mu(A) = 0$.)

   (b) For any $\epsilon > 0$, there is $\delta > 0$ such that $\lambda(A) < \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$.

UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
7 May 2005: Afternoon Session, 2:00-5:00

Name: ______________________________

6. Let U be the region in the first quadrant bounded by the unit circle and the straight line from 1 to i. Find a conformal map from U to the unit disc.

7. For any $x \in \mathbb{R}$, let I_x be the open interval, $I_x = (x-1,\ x+1)$. Suppose $E \subset \mathbb{R}$ is a Lebesgue measurable set. Define a set F by
$$F = \{x \in \mathbb{R} :\ m(I_x \cap E) > 0\}\ .$$
Prove that F is also Lebesgue measurable.

8. Compute the integral
$$\int_0^\infty \frac{\ln x}{x^2+1} dx.$$

9. (a) For a function $f \in L^1(\mathbb{R}, dx)$, a point $x \in \mathbb{R}$ is called a 'Lebesgue point' of f if
$$\lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)| dy = 0.$$
Show that if f is continuous on $\mathbb{R}$, every point on $\mathbb{R}$ is a Lebesgue point of f.

   (b) For $f \in L^1(\mathbb{R}, dx)$, its 'maximal function' $(Mf)(x)$ is defined by
$$(Mf)(x) = \lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| dy.$$
Show that
$$|f(x)| \leq (Mf)(x)$$
at every Lebesgue point of f.

10. Let $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + z_0$, $a_n \neq 0$ be a polynomial of degree exactly n. Let C be a closed contour, oriented counter-clockwise, enclosing all the roots of p_n. Compute the integral
$$\frac{1}{2\pi i} \int_C \frac{p_n'(z)}{p_n(z)} dz.$$