UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Analysis

4 May 2006: Morning Session, 9:00-12:00

1. Compute
$$\int_0^{\pi} \frac{1}{1+2\alpha\cos\theta+\alpha^2}d\theta$$
when α is real and $\alpha \neq -1, 1$.

2. Find a conformal mapping from the part of the unit disc around the origin that lies above the line from -1 to i into the part of the unit disc around the origin in the upper half plane.

3. Let $\{z_n\}_{n=1}^{\infty}$ be a discrete sequence of points in $\mathbb{C}$ with no accumulating points in $\mathbb{C}$. Show that if f is analytic in $D = \mathbb{C} \setminus \{z_1, z_2, \dots\}$ and $f(D) \subset (\mathbb{C} \setminus [1, \infty))$, then f is a constant function.

4. Suppose that $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. Show that
$$\lim_{h\to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|dx = 0.$$

5. Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that if $f \in L^p(\mu)$ for some $1 \leq p < \infty$, then
$$\lim_{t\to\infty} t^p \cdot \mu\{x : |f(x)| > t\} = 0.$$

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4 May 2006: Afternoon Session, 2:00-5:00

6. Let $f : [0, 1] \to [0, 1]$ be a function such that it is continuous, non-decreasing, $f(0) = 0$ and $f(1) = 1$. Either prove $\int_0^1 f'(x)dx = 1$ or give a counterexample.

7. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Suppose that $h(z)$ is analytic in a neighborhood of D, $h(0) = 0$ and $|h'(z)| < 1$ for all $z \in D$. Prove that the function $f(z) = z + h(z)$ is one-to-one on $\{z \in \mathbb{C} : |z| < 1\}$.

8. Compute the limit
$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx.$$

9. Prove that there is a function $f(z)$ which is analytic in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, is continuous in $\overline{D}$, and satisfies
$$\log |f(e^{i\theta})| = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n \cos^n\left(\frac{n}{2}\theta\right) \sin^n\left(\frac{n}{10}\theta\right), \qquad \theta \in [0, 2\pi).$$

10. Let $u : [0, 1] \to \mathbb{R}$ be a measurable function. Prove that there is a compact set K in $[0, 1]$ of Lebesgue measure precisely $\frac{1}{2}$ and a continuous function $g : [0, 1] \to \mathbb{R}$ such that $g|_K = u|_K$.