UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Analysis
3 September 2005: Morning Session 9:00-12:00

1. Suppose that f is an entire function such that f is real on the real axis and f is imaginary on the imaginary axis. Show that $f(z) = -f(-z)$ for all $z \in \mathbb{C}$.

2. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. For each $x \in \mathbb{R}$ find the limit

$$\lim_{s\to\infty} \sqrt{s} \int_{\mathbb{R}} h(x+t)e^{-st^2}\,dt.$$

3. (a) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of analytic functions in a region $\Omega \subset \mathbb{C}$ such that $\{f_n(z)\}$ converges for every $z \in \Omega$. Prove that $\{f_n\}$ converges uniformly on every compact subset of Ω.

   (b) Let Ω be a bounded region in $\mathbb{C}$. Suppose that f_n, $n = 1, 2, 3, \ldots$, are analytic in $\overline{\Omega}$. Prove that if $\{f_n\}$ converges uniformly on the boundary of Ω, then $\{f_n\}$ converges uniformly in $\overline{\Omega}$.

4. Let (X, μ) be a measure space such that $\mu(X) < \infty$. Show that

$$\lim_{n\to\infty} \int_X \frac{|f_n - f|}{1 + |f_n - f|}\,d\mu = 0$$

if and only if $f_n \to f$ in measure as $n \to \infty$.

5. Let $\{f_n\}$ be a sequence of measurable real-valued functions on the interval $[0, 1]$. Prove that the set of x for which $\lim_{n\to\infty} f_n(x)$ exists is measurable.

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3 September 2005: Afternoon Session 2:00-5:00

6. Let f be holomorphic in a neighborhood of the closed unit disc. Suppose that $|f(z)| < 1$ for $|z| = 1$. Show that the equation $f(z) = z$ has a unique solution in the unit disc.

7. Show that
$$\lim_{n\to\infty} \int_1^n \left(1 - \frac{t}{n}\right)^n dt = \int_1^\infty e^{-t} dt.$$

8. Let $z \in \mathbb{C} \setminus [-1, 1]$. Compute the integral
$$f(z) = \int_{-1}^{1} \frac{1}{(x-z)\sqrt{1-x^2}} dx$$
where $\sqrt{1-x^2}$ denotes the usual real square-root. In addition, express $f(i) = a + ib$ for some real a, b.

9. For a (real-valued) function f on $\mathbb{R}$, define
$$f^y(x) = f(x-y), \qquad y \in \mathbb{R}.$$
   (a) Suppose f is a continuous function on $\mathbb{R}$ with a compact support. Show that $\|f^y - f\|_{L^\infty(\mathbb{R})} \to 0$ as $y \to 0$.
   (b) Show that if $f \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$, then $\|f^y - f\|_{L^p(\mathbb{R})} \to 0$ as $y \to 0$.
   (c) Give an example of a function $f \in L^\infty(\mathbb{R})$ such that $\|f^y - f\|_{L^\infty(\mathbb{R})} \not\to 0$ as $y \to 0$.

10. Suppose that Ω is a bounded region in $\mathbb{C}$ and let $a \in \Omega$. Suppose that f is an analytic function in Ω such that $f(\Omega) \subset \Omega$ and $f(a) = a$. Set $f_1(z) = f(z)$ and $f_n(z) = f_{n-1}(f(z))$.
   (a) Compute $f_n'(a)$ in terms of $f'(a)$.
   (b) Using (a), prove that $|f'(a)| \leq 1$.