UNIVERSITY OF MICHIGAN

DEPARTMENT OF MATHEMATICS

Qualifying Review Examination in Analysis

9 September 2006: Morning Session, 9:00-12:00

1. Evaluate the integal
$$\int_0^\infty \frac{\cos x - 1}{x^2(1+x^2)} dx.$$

2. Let f be an absolutely continuous function in $(-1,1)$ such that $f(0) = 0$ and $f' \in L^2((-1,1))$. Show that
$$\lim_{x \downarrow 0} \frac{f(x)}{\sqrt{x}} = 0.$$

3. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be an analytic function in $\Delta^*(0,1) = \{z \in \mathbb{C}; 0 < |z| < 1\}$ such that $a_n \neq 0$ for an infinitely many negative indices n. Prove that for every $c \in \mathbb{C}$ and $\epsilon > 0$, $\frac{f(z)}{1-cf(z)}$ is unbounded in $\Delta^*(0,\epsilon)$.

4. Let f be an entire function in the complex plane. Suppose that there is a positive integer N such that $\frac{f(z)}{z^N} \to 0$ as $|z| \to \infty$. Prove that $f(z)$ is a polynomial of degree at most $N-1$.

5. Let $A \in [0,1]$ be a measurable set of positive measure. Prove that there are two points $x, y \in A$ such that $x \neq y$ and $x - y$ is rational. (Hint: Note that for a rational $r \in [0,1]$, $A + r \in [0,2]$.)

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9 September 2006: Afternoon Session, 2:00-5:00

6. Let $D = \{z \in \mathbb{C} : |z| < 1\} \cap \{re^{i\theta} \in \mathbb{C} : r > 0, \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$. Find a conformal mapping from D to the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

7. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions in $L^1(\mathbb{R})$ and let $f \in L^1(\mathbb{R})$. Assume that there is a constant $C > 0$ such that
$$\int_{-\infty}^{\infty} |f_n(x) - f(x)| dx \leq \frac{C}{n^2}$$
for all $n \geq 1$. Prove that $f_n \to f$ almost everywhere with respect to the Lebesgue measure.

8. Let $n \geq 2$ be an integer and let c be a complex number such that $|c| > e$. Compute the integral
$$\int_C \frac{e^z - ncz^{n-1}}{e^z - cz^n} dz$$
where $C = \{z \in \mathbb{C} : |z| = 3\}$.

9. Let
$$g(x) = \frac{1}{x^{6/7} + 1}.$$
Consider the map $T : L^3(\mathbb{R}) \to \mathbb{R}$ defined by
$$Tf = \int_{-\infty}^{\infty} f(x)g(x)dx, \qquad f \in L^3(\mathbb{R}).$$
Show that T is a linear functional on $L^3(\mathbb{R})$.

10. (a) If possible, find a nested sequence of nonempty bounded sets in $\mathbb{R}^2$ whose intersection is empty. Explain your reasoning.

    (b) If possible, do the same with nonempty closed and bounded sets. Explain your reasoning.