Department of Mathematics QR Exam Syllabus in Applied Analysis
Math 572 : Numerical Methods for Scientific Computing II

1. Ordinary Differential Equations

a) 2-point boundary value problems $y'' + c(x)y = f(x)$ with Dirichlet BC
- finite-difference approximation, $D_+D_-u_j + c_ju_j = f_j$
- Gaussian elimination for tridiagonal systems
- local truncation error, consistency
- vector and matrix norms
- stability in ∞-norm by discrete maximum principle
- derivation of eigenvalues and eigenvectors, stability in 2-norm by Fourier analysis
- consistency + stability $\Rightarrow$ convergence

b) initial value problems $y' = f(y)$
- forward Euler method
  - local truncation error, consistency + stability $\Rightarrow$ convergence
  - asymptotic expansion for the error, Richardson extrapolation
  - region of absolute stability, A-stability, backward Euler method, trapezoid method
  - application to linear systems $y' = Ay$
- Runge-Kutta methods
  - modified Euler method, midpoint method, RK4
  - order of accuracy, region of absolute stability
- general explicit 1-step methods of the form $u_{n+1} = u_n + F(u_n, h)$
  - consistency + stability $\Rightarrow$ convergence
- multistep methods
  - Adams-Bashforth, Adams-Moulton methods
    - order of accuracy, region of absolute stability, predictor-corrector methods
- theory of general multistep methods
  - linear difference equations of the form $\alpha_0 u_n + \alpha_1 u_{n-1} + \cdots + \alpha_k u_{n-k} = 0$
  - characteristic polynomials $\rho(\zeta), \sigma(\zeta)$
  - root condition $\Leftrightarrow$ stability, consistency + stability $\Rightarrow$ convergence
  - Dahlquist results on maximum order of stable, A-stable multistep schemes (statement)
- leap-frog method
  - order of accuracy, region of absolute stability, weak instability
- other examples
  - Milne's method, implicit Runge-Kutta methods, Gear's BDF methods

2. Partial Differential Equations

a) 2D Laplace equation $u_{xx} + u_{yy} = f$ with Dirichlet BC
- 5-point discrete Laplacian
- solving linear systems
  - Gaussian elimination for banded systems
  - basic iterative methods (Jacobi, Gauss-Seidel, SOR), convergence rate
- local truncation error, consistency
- stability in ∞-norm by discrete maximum principle
- derivation of eigenvalues and eigenvectors, stability in 2-norm by Fourier analysis
- consistency + stability $\Rightarrow$ convergence

b) 1D heat equation $u_t = u_{xx}$

forward/central scheme

free-space BC, Dirichlet or Neuman BC on $[0, 1]$

local truncation error, consistency

stability in ∞-norm by discrete maximum principle

amplification factor, stability in 2-norm by Fourier analysis

derivation of eigenvalues and eigenvectors for Dirichlet and Neumann BC

consistency + stability $\Rightarrow$ convergence

stability in 2-norm by discrete energy method

backward/central scheme

positive definite matrices, Cholesky factorization, stability in ∞-norm and 2-norm

Crank-Nicolson method

stability in ∞-norm and 2-norm

c) 2D heat equation $u_t = u_{xx} + u_{yy}$

forward/central, backward/central, Crank-Nicolson schemes

operator splitting

application to $y' = (A + B)y$

accuracy and stability of ADI for 2D heat equation

d) 1D scalar convection equation $u_t + cu_x = 0$

central, upwind, downwind, Lax-Friedrichs, Lax-Wendroff, leap-frog schemes

characteristics, domain of dependence, CFL condition

stability in ∞-norm, amplification factor, stability in 2-norm by Fourier analysis

consistency + stablity $\Rightarrow$ convergence

model equation, artificial viscosity, phase error, numerical wave speed

e) hyperbolic systems $u_t + Au_x = 0$

2nd order wave equation $u_{tt} = c^2 u_{xx}$ expressed as a first order system

amplification matrix, stability in 2-norm $\Leftrightarrow$ uniformly power bounded

stability of Lax-Wendroff and leap-frog methods

stability in 2-norm by energy method

von Neumann condition, Lax equivalence theorem (statement)

f) miscellaneous

lower order terms $u_t + cu_x = bu$, Strang's perturbation theorem

convection-diffusion equations $u_t + cu_x = \nu u_{xx}$

higher order dispersive equations $u_t = u_{xxx}$

References

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J.D. Lambert, Numerical Methods for Ordinary Differential Systems

K.W. Morton and D.F. Myers, Numerical Solution of Partial Differential Equations

J.W. Thomas, Numerical Partial Differential Equations

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