Problems to Keep You Sharp …

Dear Math Circlers,

Alas, Thanksgiving is upon us, and already the first season of Michigan Math Circle is over too soon.. I hope to see all of you back here on January 20th, when Thursday circles will resume.

But I know what you're thinking. What happens after we solve all the unsolved problems from Math Circle? How will we pass the time until Math Circle restarts? I think we can all agree that two months is much too long to go without some fresh math to think about. But fear not.

Here you'll find a selection of problems and challenges to keep you sharp. We know that you represent a variety of different ages and a variety of different interests and prior knowledge. You'll find quite a variety among these problems too. Some are easier, some are harder. Some are short one-shot questions, others could fill most of a Math Circle session. Some are old classics, and some have new twists. If a problem seems too intimidating to be fun, feel free to just move on to another.

And if you're *still* bored over break, the last page of this document contains some links to some online fun/math content.

Namaste.

Your faithful Keep-you-sharp-ener,
Prof. Cap Khoury

P.S. If you have any questions about anything in this document, or if you want to discuss any of these questions via email, feel welcome to email me at mjkhoury@umich.edu (please include "Math Circle" in the subject line).

Problems to Keep You Sharp ...

1. The consecutive odd numbers 3, 5, and 7 are all prime. When is the next time that three consecutive odd numbers are all prime?

2. The fractions $\frac{16}{64}$ and $\frac{19}{95}$ are very easy to simplify! (Just cancel the 6 and the 9.) Can you find any other fractions that work that way?

3. When I was your age, McDonald's sold Chicken McNuggets in three increments. One could buy 6, 9, or 20 at a time. So if you wanted 12 (6+6) or 29 (20+9) or 35 (20+9+6), no problem. On the other hand, there was no way to get exactly 17.

   a) There is a largest impossible number *N*, in the sense that it's impossible to order exactly *N* McNuggets, but any larger number can be ordered exactly. What number is that?

   b) What if they discontinue the 20-box, selling only 6 and 9 McNuggets at a time? Is there still a largest impossible number? If so, what is it?

   c) What if they discontinue the 9-box, selling only 6 and 20 McNuggets at a time? Is there still a largest impossible number? If so, what is it?

   d) What if they discontinue the 6-box, selling only 9 and 20 McNuggets at a time? Is there still a largest impossible number? If so, what is it?

   e) Suppose you have your own restaurant, and you want it to be mathematically interesting. Can you think of three numbers *a, b, c* so that if you sell your nuggets in boxes of *a*, *b*, or *c*, then there is a largest impossible number, but that if you only use any two of the numbers, there is no largest impossible number?

4. Five beginners, six masters, and two grandmasters wish to play in a chess tournament.

   a) Can you arrange a playing schedule so that each beginner plays just once, each master plays twice, and each grandmaster plays three times?

   b) Can you arrange a playing schedule so that each beginner plays just twice, each master plays three times, and each grandmaster plays four times?

5. I have a selection of four dice. Each die has six sides and is fair in the sense that each side is equally likely to turn up on any given throw. The red die has sides labelled 1,1,1,5,5,5. The white die has sides labelled 2,2,2,2,6,6. The blue die has sides labelled 3,3,3,3,3,3. The brown die has sides labelled 0,0,4,4,4,4. Two people play a game in which each rolls a

Problems to Keep You Sharp ...

die (not necessarily the same die), and the higher number wins.

a) If one player rolls red and the other rolls white, who is more likely to win? How much more likely?

b) If one player rolls blue and the other rolls white, who is more likely to win? How much more likely?

c) If one player rolls blue and the other rolls brown, who is more likely to win? How much more likely?

d) If one player rolls red and the other rolls brown, who is more likely to win? How much more likely?

e) Anything surprising about your findings?

6. "Repunits" are numbers which use only the digit 1. 1, 11, 111, 1111, 11111, and so on. Let R_n be the number with n 1s.

   a) Compute the prime factorizations of the first several R_n (say, up to R_{10}).

   b) Compare the factor lists of the various R_n . Which repunits seem to have no factors in common? Can you find a way to predict whether or not R_{15} and R_{28} have any factors in common?

   c) If you made a list of the prime factors of all the repunits, would all the prime numbers show up eventually? (Answer: yes, with two exceptions. Which exceptions?)

   d) Can you see any pattern to when prime numbers appear as factors for the first time?

   e) Repeat this exercise in a different number base.

7. A regular polygon is a polygon with equal angles and equal sides. (A regular triangle is usually called an equilateral triangle, and a regular quadrilateral is usually called a square.) Ordinary graph paper shows a way of tiling the plane with squares. You've probably also seen tilings with regular triangles and regular hexagons. Why isn't there something similar with pentagons?

8. How many three-digit numbers can you find in base 5 which are divisible by 7 and which would still be divisible by 7 if you read them base 10?

9. You are playing a game in which you throw a pair of six-sided dice and move according to the sum of the rolls (similar to Monopoly, except that we don't care about doubles). You

Problems to Keep You Sharp …

have a pair of fair dice which are not yet labelled. You can put any number of dots you choose on each side of each die. Is it possible to make a "custom" pair of dice such that every roll from 2 to 12 is just as likely on your dice as it would be on standard dice. If you found a solution, is it the only one?

10. How many 3-letter "words" are there consisting of only the letters X and Y, such that two X's never appear in a row? How many 4-letter words? 5-letter? 10-letter? 20-letter? Your answers should look familiar. Is that just a coincidence? (Hint: no.)

11. How many 3-letter "words" are there consisting of only the letters A, B, and C, such that a C never appears immediately after an A? How many 4-letter words? 5-letter? 10-letter? This question and the previous question have closely related answers. Is that just a coincidence? (Hint: no.)

12. How many 3-letter "words" are there consisting of only the letters D, E, F, G, H, such that the clusters DD, DE, and GD never occur? How many 4-letter words? 5-letter? 10-letter? This question and the previous questions have closely related answers. Is that just a coincidence? (Hint: no.)

13. (Quite hard.) What question would continue the pattern of the previous three questions?

14. A group of 19 people is having a ping-pong tournament. Each person will compete against each other person exactly once, and there are no ties. Each person makes a list of all the people she beat, and a list of all the people beaten by someone she beat. If these lists together include all the other people, that person is awarded a pizza.

    a) Must there always be at least one pizza awarded? Why or why not?

    b) Is it possible for a person to lose some of her games but still be the only person who gets a pizza?

15. This question is also about a similar ping-pong tournament, with the same rules as in the previous question. If it's possible to everyone in the tournament to stand in a circle so that each person beat the person on her left and lost to the person on her right, then we say the tournament is "circle-friendly". If it's possible to divide everyone into two groups, red and blue, at least one person in each group, so that each person in the red group beat each person in the blue group, then we say the tournament is "group-friendly".

    a) Can you find a tournament which is both circle-friendly and group-friendly? Why or why not?

Problems to Keep You Sharp …

b) (Harder.) Can you find a tournament which is neither circle-friendly not group-friendly? Why or why not?

16. You're probably familiar with base 10, and base 2, and other similar bases. There are such things as "negative bases", such as base -10 or -5. In base -5, the digits are 0,1,2,3,4; the place values are "one", "negative five", "twenty-five", "negative one hundred twenty-five", etc.

    a) Can you figure out how to write all the numbers from 1 to 100? What about negative numbers, from -1 down to -100? (Hint: you don't use negative signs.)

    b) Can you work out the rules for addition and subtraction? (Carrying and borrowing may be tricky to adapt.)

    c) (Harder.) Can you work out the rules for multiplication?

17. We could also use base 5 (positive five, not negative), but change what we allow for digits. Instead of 0,1,2,3,4, what if we use -2,-1,0,1,2?

    a) Can you figure out how to write all the numbers from 1 to 100? What about negative numbers, from -1 down to -100?

    b) Can you work out the rules for addition and subtraction? (Carrying and borrowing may be tricky to adapt.)

    c) (Harder.) Can you work out the rules for multiplication?

18.

    a) Is it possible to color the squares in a 5x5 grid with two colors so that no four squares of the same color form a rectangle?

    b) Is it possible to color the squares in a 4x6 grid with two colors so that no four squares of the same color form a rectangle?

    c) Is it possible to color the squares in a 3x7 grid with two colors so that no four squares of the same color form a rectangle?

19. Imagine an infinite sheet of graph paper. Can you find three lattice points which form an equilateral triangle? Why or why not?

20. Imagine an infinite "honeycomb" made of regular hexagons. Can you find four vertices of hexagons somewhere in the grid that make a perfect square? Why or why not?

Problems to Keep You Sharp …

21. Consider the following process. Starting with any positive integer, we form a new integer by squaring each digit and adding the results. For example, if we start with 2, we reach 4. then 16, then 37, then 58, then 89, then 145. then 42, then 20, then back to 4; from this point on, the cycle will repeat. If we start with 1, on the other hand, we remain at the number 1.

    a) Repeat the process starting with 3, 4, 5, and some other small numbers. Also try a few larger numbers. Do all your examples either reach 1 or the cycle containing 4?

    b) Is there any way to be sure that you've found all the possible ending cycles? Are there are any starting numbers that don't lead to a cycle?

    c) What if, instead of squaring each digit, we cube each digit? Take higher powers?

    d) What if we ask the same question in a different number base?

22. You have a small selection of weights and a balance scale. You can measure out specific weights by placing some or all of the weights on one side and the object to be measured on the other side. For example, if you had weights measuring 3, 4, and 9 ounces, you could use them to measure out 3, 4, 7, 9, 12, 13, or 16 ounces.

    a) Suppose you want to be able to measure any whole number weight from 1 to 25 ounces. Can you find a set of five weights that will accomplish this?

    b) Suppose that you can have four weights of your choice, and you want to be able to measure any whole number weight from 1 to N, where N is as large as possible. What weights should you choose? How big can you make N?

    c) What if you have six weights? seven? k?

23. Let's make a small change to the previous problem. You are now allowed to put weights on both sides of the scale. (In practice, this means you can subtract as well as add.) For example, if you had weights measuring 3, 4, and 9 ounces, you could use them to measure out 3, 4, 7, 9, 12, 13, or 16 ounces as before, but also 1, 2, 5, and 6 ounces.

    a) Suppose you want to be able to measure any whole number weight from 1 to 25 ounces. Can you find a set of FOUR weights that will accomplish this?

    b) Suppose that you can have five weights of your choice, and you want to be able to measure any whole number weight from 1 to N, where N is as large as possible. What weights should you choose? How big can you make N?

Problems to Keep You Sharp ...

c) What if you have six weights? seven? k?

24. The numbers 1, 2, 3, 5, 8, 13, 21, 34, ... are called Fibonacci numbers (to continue the sequence, you repeatedly add the previous two numbers to get the next).

   a) Can you write the number 19 as a sum of distinct Fibonacci numbers (that is, you can't repeat numbers as in 8+8+3 or 13+3+3? How many different ways can you find to do this?

   b) What about the number 100?

   c) Is it possible to write every positive integer as a sum of Fibonacci numbers?

   d) Suppose we add a new rule that you can't use two consecutive Fibonacci numbers in the same sum. Is it still possible to write every positive integer under this new rule?

   e) Is it ever possible to obtain the same number in two different ways under the new rule?

25. Given a sequence of numbers, we can form the "difference sequence" by subtracting consecutive terms. For example, suppose we start with the sequence of squares 0,1,4,9,16,25,36,49,...; the difference sequence is 1,3,5,7,9,11,... . If we repeat the process, we get the second difference sequence, 2,2,2,2,2,...; notice that all the “second differences” are the same.

   a) Can you find more sequences which become constant after you take the second differences? Can you find ALL such sequences?

   b) Can you find a sequence which is the same as its difference sequence? Can you find ALL such sequences?

   c) Can you find a sequence such that the difference sequence is the same as twice the original?

26. There are four one-digit numbers which, when squared, end in the original digit: $0^2=0, 1^1=1, 5^2=25, 6^2=36$.

   a) How many two-digit numbers are there that, when squared, end in the original number?

   b) How many three-digit numbers are there that, when squared, end in the original number?

   c) How many four-digit numbers are there that, when squared, end in the original number?

   d) How many ten-digit numbers do you think have the same property? (Notice that trial

Problems to Keep You Sharp …

and error/brute force is getting less and less helpful.)

e) Can you find the ten-digit numbers?

27. If n is a positive integer, then *n factorial* is defined by $n! = 1 \times 2 \times 3 \times 4 \times \cdots \times n$.

   a) How many zeroes are at the end of 100! ?

   b) Are there any numbers *n* such that *n!* ends in exactly 100 zeroes?

28. How many digits does $8675309^{8675309}$ have? (You can and should use a calculator, but you don't need a supercomputer. A cheap scientific calculator is good enough, if you're doing it right.)

29. A palindrome is a number which remains the same if you reverse the order of its digits.

   a) Can you find a five-digit palindrome which is divisible by 11? If not, why not?

   b) Can you find a six-digit palindrome which **isn't** divisible by 11? If not, why not?

   c) Which are there more of? Twenty-digit palindromes or nineteen-digit palindromes?

30. Can't hurt to end on a classic. A certain school has 1000 students and 1000 lockers (numbered, naturally, from 1 to 1000, in order down a single long hallway). At the start of the day, all the lockers are closed. Then each student walks the hall in turn, from student 1 to student 1000. First, student 1 opens all the lockers. Next, student 2 closes lockers 2,4,6,8,10,...,1000, Then student 3 closes locker 3, opens locker 6, closes locker 9, opens locker 12, etc. In general, student k "switches" every kth locker, opening it if is closed and closing it if is opened.

   a) At the end of the day, how many lockers will be open?

   b) Suppose now that the students do not necessarily walk the hall in numerical order. Each student will still walk the hall exactly once and, no matter when she takes her turn, student #k switches every kth locker (opening it if is closed and closing it if is opened). What is the largest possible number of open lockers at the end of the day? What is the smallest possible number of open lockers at the end of the day?

Problems to Keep You Sharp …

.(...where by problems, here I mean "links".)

Math Podcasts:

There are more and more math-related podcasts (and blogs) these days, at all levels. It's a good time to be a math enthusiast. Here are two places to start.

- http://www.aracnet.com/~eseligma/mm/ If you have three minutes and you want to hear something neat, you can count on Math Mutation. Each podcast has a transcript and references.
- http://mathfactor.uark.edu/ Math Factor has longer pieces, and as a result they can go a bit deeper.

Miscellaneous Math Sites:

Just a handful of selected sites to get lost in some interesting mathematics.

- http://www.math.hmc.edu/funfacts/ Mudd Math Fun Facts
- http://euler.slu.edu/escher/index.php/Math_and_the_Art_of_M._C._Escher EscherMath, a huge and amazing collection of information on mathematics underlying Escher's art.
- http://www.dimensions-math.org/ A series of films leading ultimately to the fourth dimension.

Puzzles and Games:

Of course, exploring a new math idea (or an old one from a new direction) is one of the best kinds of play, so it shouldn't be a surprise that so many "official" kinds of games and puzzles have connections to mathematics. Here are some of my favorite sources of puzzles with higher-than-average levels of mathematical structure and subtext.

- http://geometrygames.org/TorusGames/index.html Play familiar games in an unfamiliar setting.
- http://logicmazes.com/ Robert Abbott's website for mazes that are more interesting than you think. (The hardest? "Where Are the Cows?", but don't try that one until you've properly warmed up. It took me a week to solve.)
- http://www.puzzlebeast.com/ Puzzle Beast, a huge collection of puzzles with an interesting backstory. My favorites are Fried Okra, Dry Cleaning, and Meandering Monks. (Advice: brute force trial and error won't get you very far.)
- http://motris.livejournal.com/ The Art of Puzzles, the blog of Thomas Snyder, champion solver and master craftsman of sudoku and other puzzles. For his thoughts on puzzles (which are often mathematical, just under the surface), read the blog. Each Friday he posts a puzzle of some kind (usually hard, always pretty); if you want to start somewhere, try his The Art of Sudoku puzzles.
- http://www.superliminal.com/cube/cube.htm Are ordinary Rubik's Cube's too mundane? **Try a four-dimensional version.**
- http://www.gravitation3d.com/magiccube5d/ Was that too easy? Have a ***five-dimensional*** Rubik's cube. (I can't do this one.)