UM RTG Lecture Series: Heegaard Floer meets Seiberg–Witten

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JOINT WORK WITH YI-JEN LEE AND CLIFFORD H. TAUBES

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Let X be a compact, connected, and smooth manifold of dimension n, and $f : X \to \mathbb{R}$ be a smooth function.

Fix a Riemannian metric $\mathfrak{g}$ on X. The gradient vector field ∇f of f is defined by $\mathfrak{g}(\nabla f, \cdot) = df$.

A critical point $p \in X$ of f, i.e. $df_p = 0$, is called *non-degenerate* if $D_p \nabla f : T_p X \to T_p X$ (Hessian) is an isomorphism.

f is called a *Morse* function if all its critical points are non-degenerate.

Let f be a Morse function. Then,

$$-\nabla f \rightsquigarrow \varphi_s,$$

For a critical point p of f, define

$$\begin{aligned}\mathcal{A}(p) &:= \{x \in X \mid \lim_{s\to\infty} \varphi_s(x) = p\},\\ \mathcal{D}(p) &:= \{x \in X \mid \lim_{s\to-\infty} \varphi_s(x) = p\}.\end{aligned}$$

Let $index(p) := dim(\mathcal{D}(p))$.

Morse–Smale condition: For any pair (p, q) of critical points of f, we have $\mathcal{D}(p) \pitchfork \mathcal{A}(q)$.

Under the Morse–Smale condition $\mathcal{M}(p, q) := \mathcal{D}(p) \pitchfork \mathcal{A}(q)$ is a smooth manifold of dimension

$$index(p) - index(q),$$

when $index(p) \geq index(q)$.

Orientation of $\mathcal{M}(p, q)$: A choice of orientation for each $\mathcal{D}(p)$ induces an orientation on each $\mathcal{M}(p, q)$ via

$$0 \to T_x\mathcal{M}(p, q) \hookrightarrow T_x\mathcal{D}(p) \twoheadrightarrow N_x\mathcal{A}(q) \to 0,$$

after an appeal to the fact that $N_x\mathcal{A}(q) \simeq T_x\mathcal{D}(q)$.

The Morse–Witten complex:

- Chain group

$$C_* := \mathbb{Z}\langle p \in X \mid df_p = 0 \rangle.$$

- Grading

  Defined by the indices of critical points. The relative grading is defined by

$$gr(p,q) = index(p) - index(q) = \text{Spectral flow of the Hessian.}$$

- Differential

  Negative gradient trajectories,

$$\frac{d}{ds} x(s) = -\nabla f(x(s)).$$

  Then

$$\partial_* p = \sum_{gr(p,q)=1} \#(\mathcal{M}(p,q)/\mathbb{R}) q.$$

HEEGAARD FLOER HOMOLOGY

Peter Ozsváth & Zoltan Szabó

THE SETUP

$f : M \to \mathbb{R}$ self-indexing Morse function with

- One maximum and one minimum,
- g pairs of index-1 and index-2 critical points.

Fix a pseudo-gradient vector field $\mathfrak{v}$ for f.
$\rightsquigarrow (\Sigma, \alpha, \beta)$ a Heegaard diagram;

- Σ closed, connected, oriented genus-g surface,
- $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ and $\beta = \{\beta_1, \ldots, \beta_g\}$.

Let $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$, and fix $z \in \Sigma \setminus (\alpha \cup \beta)$.

$$\text{A Spin}^{\mathbb{C}}\,\text{structure on M} \rightsquigarrow \mathfrak{S} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta.$$

- Chain groups

$$\begin{aligned} CF^{\infty}(M,\mathfrak{s}) &:= \mathbb{Z}\langle [x,i] \,|\, x \in \mathfrak{S},\ i \in \mathbb{Z}\rangle, \\ CF^{-}(M,\mathfrak{s}) &:= \mathbb{Z}\langle [x,i] \,|\, x \in \mathfrak{S},\ i \in \mathbb{Z}\ i<0\rangle, \\ CF^{+}(M,\mathfrak{s}) &:= CF^{\infty}(M,\mathfrak{s})/CF^{-}(M,\mathfrak{s}). \end{aligned}$$

- Relative grading
  $x, y \in \mathfrak{S}$ and $\phi \in \pi_2(x,y) \simeq H_2(M;\mathbb{Z}) \oplus \mathbb{Z}$ (for $g > 1$),

$$gr([x,i],[y,j]) = \mu(\phi) - 2n_z(\phi) + 2(i-j).$$

  Well defined modulo $d := gcd\{\langle c_1(\mathfrak{s}), \sigma\rangle \,|\, \sigma \in H_2(M;\mathbb{Z})\}$.
- Differential

$$\partial^{\infty}[x,i] = \sum_{\phi\in\pi_2(x,y)\ ,\ \mu(\phi)=1} \#(\mathcal{M}(\phi)/\mathbb{R})[y, i-n_z(\phi)].$$

There is a degree -2 map:

$$U : CF^\infty(M,\mathfrak{s}) \longrightarrow CF^\infty(M,\mathfrak{s}),$$

defined by $U[x,i]=[x,i-1]$, and commutes with ∂^∞.
The short exact sequence

$$0 \to CF^-(M,\mathfrak{s}) \stackrel{i}{\hookrightarrow} CF^\infty(M,\mathfrak{s}) \stackrel{\pi}{\twoheadrightarrow} CF^+(M,\mathfrak{s}) \to 0$$

yields a long-exact sequence of $\mathbb{Z}[U]$-modules

$$\cdots \to HF^-(M,\mathfrak{s}) \xrightarrow{i_*} HF^\infty(M,\mathfrak{s}) \xrightarrow{\pi_*} HF^+(M,\mathfrak{s}) \to \cdots .$$

For example,

$$\begin{aligned} HF^-(S^3) &\simeq \mathbb{Z}[U], \\ HF^\infty(S^3) &\simeq \mathbb{Z}[U,U^{-1}], \\ HF^+(S^3) &\simeq \mathbb{Z}[U,U^{-1}]/\mathbb{Z}[U]. \end{aligned}$$

seiberg–witten floer homology

Peter Kronheimer & Tom Mrowka

Fix a Riemannian metric $\mathfrak{g}$ and a $\text{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}$ on M:

- $\mathbb{S} \to M$, a Hermitian $\mathbb{C}^2$-bundle (Spinor bundle).
- $\mathfrak{cl} : T^*M \to \text{End}_{\mathbb{C}}(\mathbb{S})$, an isometry onto traceless, skew-Hermitian endomorphisms (Clifford multiplication).

Let

- $\mathbb{A}$ be a $\text{Spin}^{\mathbb{C}}$ connection on $\mathbb{S}$, i.e. a unitary connection that satisfies the Leibniz rule with respect to $\mathfrak{cl}$.
- ψ be a smooth section of $\mathbb{S} \to M$,

and denote by $\mathcal{C}$ the space of pairs of the form $(\mathbb{A}, \psi)$.

Then,

- Chern–Simons–Dirac functional

$$\mathfrak{csd}(\mathbb{A},\psi) := -\frac{1}{8}\int_M (\mathbb{A}^t - \mathbb{A}_0^t)\wedge(F_{\mathbb{A}^t} + F_{\mathbb{A}_0^t}) + \frac{1}{2}\int_M \langle \psi, \mathcal{D}_{\mathbb{A}}\psi\rangle.$$

- Seiberg–Witten equations
  Via L^2-gradient of $\mathfrak{csd}$,

$$\begin{aligned} \frac{1}{2} * F_{\mathbb{A}^t} &= \psi^t \tau \psi, \\ \mathcal{D}_{\mathbb{A}}\psi &= 0. \end{aligned}$$

- Action of gauge group $\mathcal{G} = C^\infty(M; S^1)$

$$u\cdot(\mathbb{A},\psi) := (\mathbb{A} - u^{-1}du \otimes 1_{\mathbb{S}}, u\psi).$$

  A configuration $(\mathbb{A},\psi)$ is called *reducible* if $\psi = 0$. Otherwise, it is called *irreducible*.

$$\mathcal{C} \underset{/\mathcal{G}}{\overset{blow\ up}{\rightsquigarrow}} \widetilde{\mathcal{B}}$$

$\widetilde{\mathcal{B}}$ is a manifold with boundary. The L^2-gradient vector field of $\mathfrak{csd}$ on $\mathcal{C}$ induces a vector field on $\widetilde{\mathcal{B}}$ tangent to $\partial\widetilde{\mathcal{B}}$. The points where the latter vanishes fall into three types, and

- Chain groups

$$\begin{aligned} \overline{C}_* &:= C^s \oplus C^u, \\ \check{C}_* &:= C^o \oplus C^s, \\ \widehat{C}_* &:= C^o \oplus C^u. \end{aligned}$$

- Relative grading = Spectral flow
  Well defined modulo

$$d := gcd\{\langle c_1(\mathfrak{s}), [-\frac{i}{2\pi} u^{-1} du] \,|\, u \in C^\infty(M, S^1)\}.$$

- Differential
  Via trajectories in $\widetilde{\mathcal{B}}$.

There is a degree -2 map:

$$U : HM_*(M, \mathfrak{s}) \longrightarrow HM_*(M, \mathfrak{s}),$$

defined by the action of the generator of $H^2(\widetilde{\mathcal{B}}; \mathbb{Z})$.

There exists an exact sequence

$$\cdots \to \widehat{HM}_*(M, \mathfrak{s}) \xrightarrow{p_*} \overline{HM}_*(M, \mathfrak{s}) \xrightarrow{i_*} \widecheck{HM}_*(M, \mathfrak{s}) \xrightarrow{j_*} \widehat{HM}_*(M, \mathfrak{s}) \to \cdots$$

of graded $\mathbb{Z}[U]$-modules.

(Analogous to the singular homology of a pair $(X, \partial X)$;

$$\cdots \to H_{n+1}(X, \partial X) \xrightarrow{p_*} H_n(\partial X) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, \partial X) \to \cdots)$$

	$\cdots$	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\cdots$
$\widecheck{HM}_*(S^3)$	$\cdots$	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
$\widehat{HM}_*(S^3)$	$\cdots$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0	0	0	$\cdots$
$\overline{HM}_*(S^3)$	$\cdots$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$

Table: Seiberg–Witten Floer homology of S^3.

PERTURBATIONS OF $\mathfrak{csd}$

Perturb the Chern–Simons–Dirac functional by

$$-\frac{1}{2}\int_M (\mathbb{A}^t - \mathbb{A}_0^t) \wedge iw,$$

where w is a closed 2-form. Then,

$$\mathfrak{csd}(u \cdot (\mathbb{A}, \psi)) - \mathfrak{csd}(\mathbb{A}, \psi) = [-\frac{i}{2\pi}u^{-1}du] \cdot c,$$

where $c = 2\pi(\pi c_1(\mathfrak{s}) - [w])$, called the *period* of the perturbed $\mathfrak{csd}$.

- $c = 0$ in $H^2(M; \mathbb{R})$ (Balanced).
- $c \neq 0$ in $H^2(M; \mathbb{R})$, suppose $c = \lambda c_1(\mathfrak{s})$ (Monotone).

Main Theorem

Let $\mathfrak{s}$ be a Spin$^{\mathbb{C}}$ structure on M. There exists a commutative diagram

$$\begin{array}{ccccc}
\cdots HF^-(M,\mathfrak{s}) & \longrightarrow & HF^\infty(M,\mathfrak{s}) & \longrightarrow & HF^+(M,\mathfrak{s})\cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \widehat{HM}_*(M,\mathfrak{s},c_b) & \longrightarrow & \overline{HM}_*(M,\mathfrak{s},c_b) & \longrightarrow & \widecheck{HM}_*(M,\mathfrak{s},c_b)\cdots
\end{array}$$

where the vertical arrows are isomorphisms of graded Abelian groups, while the top and the bottom rows are the respective long-exact sequences for Heegaard Floer homology and Seiberg–Witten Floer homology.

Proof of the Main Theorem (Schematically)

Let $Y = M\#_{g+1}S^1 \times S^2$ and $\overline{Y} = -M\#_{g+1}S^1 \times S^2$.

$$\begin{array}{ccc} \textit{SW Floer homology of } Y & \leftrightarrow & \textit{embedded contact homology of } \overline{Y} \\ \updownarrow & & \updownarrow \\ \textit{SW Floer homology of } M & \simeq & \textit{Heegaard Floer homology of } M \end{array}$$

Definition

- A contact structure ξ on an oriented 3-manifold M is a totally non-integrable cooriented 2-plane field.
- A contact form on M is a 1-form λ such that $\lambda \wedge d\lambda > 0$. λ is compatible with ξ if $\xi = Ker(\lambda)$.

Example

The standard contact structure ξ_{std} on $\mathbb{R}^3$ is defined by the kernel of the 1-form $dz - ydx$. (Kernel is generated by $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\}$)

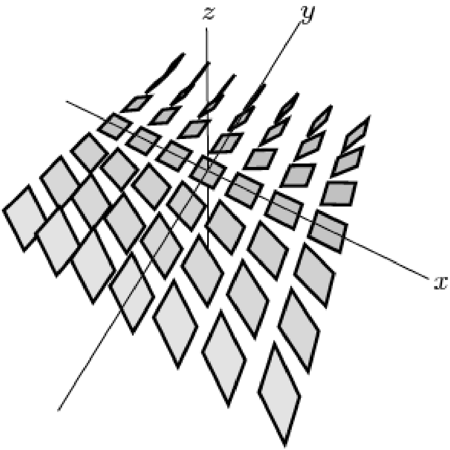

Figure: Standard contact structure on $\mathbb{R}^3$

Definition

Given a contact structure ξ on M and a compatible contact 1-form λ, the associated *Reeb* vector field R is defined by

- $d\lambda(R, \cdot) = 0$,
- $\lambda(R) = 1$.

Theorem (Taubes 2006)

Let ξ be a contact structure on a closed, oriented 3-manifold M and λ be a compatible contact 1-form. Given $\Gamma \in H_1(M; \mathbb{Z})$, there exists a finite set $\Theta = \{(\gamma, m)\}$ where γ is a periodic orbit of the Reeb vector field associated to λ and $m > 0$ is an integer such that

$$\Gamma = \sum_{(\gamma,m)\in\Omega} m[\gamma].$$

Theorem (Taubes 2008)

Let ξ *be a contact structure on a closed, oriented* 3*-manifold* M *and* λ *be a compatible contact* 1*-form. Given* $\Gamma \in H_1(M;\mathbb{Z})$*, there exists an isomorphism*

$$ECH(M,\lambda,\Gamma) \cong \widecheck{HM}_*(-M, \mathfrak{s}_\xi \otimes PD(\Gamma)).$$

Corollary

$\bigoplus_{\Gamma \in H_1(M;\mathbb{Z})} ECH(M,\lambda,\Gamma)$ *is an invariant of* M*.*

Corollary (of Main Theorem)

Let ξ be a contact structure on a closed, oriented 3-manifold M and λ be a compatible contact 1-form. Given $\Gamma \in H_1(M;\mathbb{Z})$, there exists an isomorphism

$$ECH(M, \lambda, \Gamma) \cong HF^+(-M, \mathfrak{s}_\xi \otimes PD(\Gamma)).$$

Moreover, the above isomorphism identifies the contact elements on both sides.

Remark

V. Colin, P. Ghiggini, and K. Honda announced an alternative approach to proving the above isomorphism.

Proof of the Main Theorem

Lipshitz's cylindrical reformulation of Heegaard Floer homology:

- Generators

  $[[1,2] \times x, i]$ where $x \in \mathfrak{S} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \in \mathbb{Z}$.

- Differential

  Fix an almost complex structure J on $\mathbb{R} \times [1,2] \times \Sigma$ satisfying

  1. $\omega = ds \wedge dt + w_\Sigma$ tames J, i.e. $\omega(v, Jv) > 0$ for all $v \neq 0$,
  2. J is split away from $\mathbb{R} \times [1,2] \times \alpha \cup \beta$,
  3. J is invariant under translations along $\mathbb{R}$ direction,
  4. $J\partial_s = \partial_t$ (s for the $\mathbb{R}$ factor, t for the $[1,2]$ factor),
  5. J preserves $T\Sigma$.

  Count J-holomorphic submanifolds of index 1 and their intersections with $\mathbb{R} \times [1,2] \times \{z\}$.

Given a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$,

$$\begin{array}{rcl} \text{Periodic domains} & \rightsquigarrow & H_2(M, \mathbb{Z}) \\ \mathcal{P} & \mapsto & \mathcal{H}(\mathcal{P}). \end{array}$$

Lemma

Given a Spin$^{\mathbb{C}}$ *structure* $\mathfrak{s}$*, a pointed Heegaard diagram is strongly* $\mathfrak{s}$*-admissible only if there exists an area form* w_Σ *on* Σ *such that*

- $\int_\Sigma w_\Sigma = 2$,
- $\int_{\mathcal{P}} w_\Sigma = \langle c_1(\mathfrak{s}), \mathcal{H}(\mathcal{P}) \rangle$ *for each periodic domain* $\mathcal{P}$.

Now, fix a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_M$ on M and a strongly $\mathfrak{s}_M$-admissible pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for M.

A VERSION OF MICHAEL HUTCHINGS'S ECH

Let Λ denote a pairing between the index-1 and index-2 critical points of the self-indexing Morse function f.

Let M_δ denote the 3-manifold obtained from M by excising small radius Euclidean balls around each critical point of f.

Given $\mathfrak{p} \in \Lambda$ attach a copy of $[-1,1] \times S^2$ along the boundary spheres corresponding to the critical points determined by $\mathfrak{p}$, and denote it by $\mathcal{H}_{\mathfrak{p}}$.

Attach a single copy $[-1,1] \times S^2$ along the boundary spheres corresponding to the maximum and the minimum points of f. Denote the latter by $\mathcal{H}_0$.

The resulting manifold is $M \#_{g+1} S^1 \times S^2$.

w_Σ on Σ is used to construct a *stable Hamiltonian structure* (a, w) on $\overline{Y}$ ($da = hw$ for $h : \overline{Y} \to \mathbb{R}$ smooth, $dw = 0$, and $a \wedge w > 0$) such that

- $w|_{S^2}$ in $\mathcal{H}_0$ and $w|_{f^{-1}(c)}$ for regular $c \in \mathbb{R}$ are area forms,
- $\langle [w], [S^2] \rangle = \begin{cases} 2 & \textit{on } \mathcal{H}_0 \\ 0 & \textit{on each } \mathcal{H}_\mathfrak{p} \end{cases}$
- $\langle [w], F \rangle = \langle c_1(\mathfrak{s}_M), F \rangle$ for each $F \in H_2(M; \mathbb{Z})$.

Also, a closed nowhere vanishing 1-form $\hat{a}$ on $\overline{Y}$ such that $\hat{a} \wedge w > 0$.
$\rightsquigarrow K^{-1} = ker(\hat{a})$ oriented by w such that

$$\langle e_{K^{-1}}, [S^2] \rangle = \begin{cases} 2 & \textit{on } \mathcal{H}_0 \\ -2 & \textit{on each } \mathcal{H}_\mathfrak{p} \end{cases}$$

and a canonical Spin$^\mathbb{C}$ structure $\mathfrak{s}_0$ on $\overline{Y}$.
Now, fix $\Gamma \in H_1(\overline{Y}; \mathbb{Z})$ such that $\langle PD(\Gamma), [S^2] \rangle = \begin{cases} 0 & \textit{on } \mathcal{H}_0 \\ 1 & \textit{on each } \mathcal{H}_\mathfrak{p} \end{cases}$,
and $\mathfrak{s}|_M = \mathfrak{s}_M$ where

$$\mathfrak{s} = \mathfrak{s}_0 \otimes PD(\Gamma).$$

Consider the associated *Reeb* vector field R defined by

- $w(R, \cdot) = 0$,
- $a(R) = 1$.

Then, for the *ech* chain complexes

- Generators
  (Θ, i) where $i \in \mathbb{Z}$ and $\Theta := \{(\gamma, m)\}$ such that
  1. γ is a periodic orbit of R,
  2. $m \in \mathbb{Z}$ and $m \geq 1$ with equality if γ is hyperbolic,
  3. $\Gamma = \sum m[\gamma]$.

Proposition

The periodic orbits of R that appear in finite collections from above have empty intersection with $\mathcal{H}_0$ and $M_\delta \setminus f^{-1}(1, 2)$. Moreover, they are all hyperbolic.

There is a distinguished periodic orbit γ_z of R:

- Intersects each cross-sectional S^2 in $\mathcal{H}_0$ exactly once,
- Intersects Σ at z.

Figure: The handle $\mathcal{H}_\mathfrak{p}$ regarded as a spherical shell.

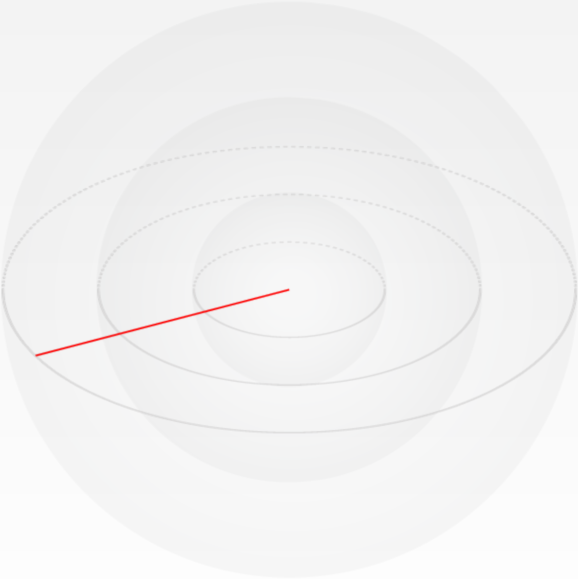

Figure: Integral curves in $\mathcal{H}_{\mathfrak{p}}$ with $\theta = \frac{\pi}{2}$.

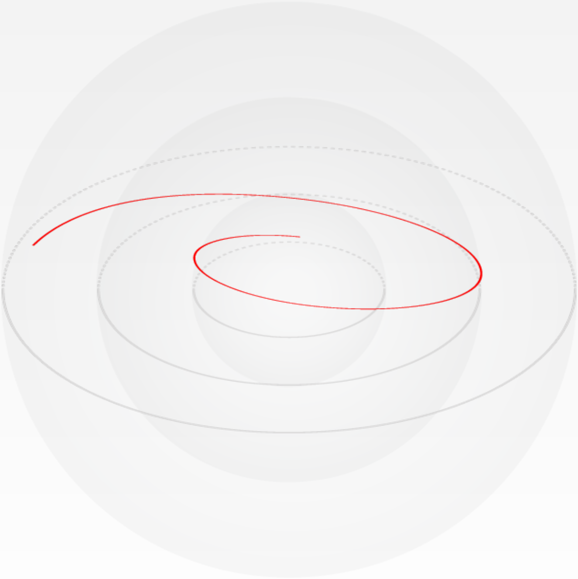

Figure: Integral curves in $\mathcal{H}_{\mathfrak{p}}$ with $0 < \cos^2\theta < \frac{1}{3}$.

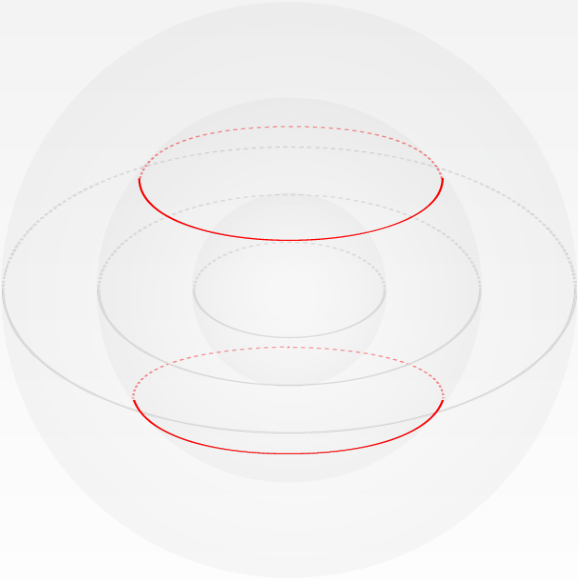

Figure: The integral curves $\gamma_{\mathfrak{p}}^{+}$ and $\gamma_{\mathfrak{p}}^{-}$ on the $\{0\} \times S^2$ slice in $\mathcal{H}_{\mathfrak{p}}$.

Fix an almost complex structure J on $\mathbb{R} \times \overline{Y}$ satisfying:

1. J is invariant under translations along $\mathbb{R}$ direction,
2. J is invariant under translations along ϕ coordinate in $\mathcal{H}_0$ and $\mathcal{H}_{\mathfrak{p}}$'s,
3. $J\partial_s = R$ (s denotes the coordinate on the $\mathbb{R}$ factor),
4. J preserves K^{-1}, and $J|_{K^{-1}}$ is compatible with $w|_{K^{-1}}$,

and some other additional desired properties.

- Relative grading
  Via *ECH index*, I, defined by M. Hutchings.

$\mathbb{R} \times (\overline{Y} \setminus \bigcup_{\mathfrak{p}\in\Lambda} \gamma_{\mathfrak{p}}^{+} \cup \gamma_{\mathfrak{p}}^{-})$ is foliated by

1. S^2's in $\mathcal{H}_o$, unobstructed and index $= 2$.
2. $f^{-1}(c) \cong \Sigma$ for $c \in (1, 2)$, obstructed and index $= 2 - 2g$.
3. $2g$-times punctured spheres with only negative ends labeled by $\bigcup_{\mathfrak{p}\in\Lambda} \gamma_{\mathfrak{p}}^{+} \cup \gamma_{\mathfrak{p}}^{-}$, obstructed and index $= 2 - 2g$.
4. Two disks with only positive ends at either $\gamma_{\mathfrak{p}}^{+}$ or $\gamma_{\mathfrak{p}}^{-}$, unobstructed and index $= 1$.
5. Cylinders with only positive ends at $\gamma_{\mathfrak{p}}^{+}$ and $\gamma_{\mathfrak{p}}^{-}$, obstructed and index $= 0$.

Lemma

Those J-holomorphic subvarieties with no irreducible components from (1) or (2)

- *has empty intersection with* $\mathbb{R} \times \mathcal{H}_0$ *and* $\mathbb{R} \times (M_\delta \setminus f^{-1}(1,2))$,
- *look very much like Lipshitz submanifolds in* $\mathbb{R} \times f^{-1}(1,2)$.

- Differential

  The *ech* differential ∂^∞_{ech} counts J-holomorphic subvarieties in $\mathbb{R} \times \overline{Y}$ with $I = 1$, and their intersections with $\mathbb{R} \times \gamma_z$.

Theorem

$\partial^\infty_{ech} = \partial^\infty_{HF} + \sum_{\mathfrak{p} \in \Lambda} \partial_*$.

∂_* is given by the rule:

- $\partial_*(m,0) = 0$ for each $m \in \mathbb{Z}$,
- $\partial_*(m,1) = (m,0) + (m+1,0)$ for each $m \in \mathbb{Z}$,
- $\partial_*(m,-1) = (m,0) + (m-1,0)$ for each $m \in \mathbb{Z}$,
- $\partial_*(m,\{1,-1\}) = (m,-1) - (m,1) + (m+1,-1) - (m-1,1)$ for each $m \in \mathbb{Z}$.

The homology of the chain complex $(\mathbb{Z}[\mathbb{Z} \times \mathrm{O}], \partial_*)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (denoted by $\hat{\mathrm{V}}$). The elements $(0,0)$ and $(0,1) - (1,-1)$ are closed and they generate the homology.

Hence, we have

Theorem (ech/HF correspondence)

There exists a commutative diagram

$$\begin{array}{ccccc}
\cdots ech^{-}(\overline{Y},\Gamma) & \longrightarrow & ech^{\infty}(\overline{Y},\Gamma) & \longrightarrow & ech^{+}(\overline{Y},\Gamma)\cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots HF^{-}(M,\mathfrak{s})\otimes \hat{\mathrm{V}}^{\otimes g} & \longrightarrow & HF^{\infty}(M,\mathfrak{s})\otimes \hat{\mathrm{V}}^{\otimes g} & \longrightarrow & HF^{+}(M,\mathfrak{s})\otimes \hat{\mathrm{V}}^{\otimes g}\cdots
\end{array}$$

where the vertical arrows are isomorphisms and both rows are long-exact sequences. All homomorphisms preserve the relative gradings and the respective module structures.

Theorem (ech/HM correspondence)

There exists a commutative diagram

$$\begin{array}{ccccc}
\cdots ech^{-}(\overline{Y},\Gamma) & \longrightarrow & ech^{\infty}(\overline{Y},\Gamma) & \longrightarrow & ech^{+}(\overline{Y},\Gamma)\cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots H_*^{-}(Y,\mathfrak{s}) & \longrightarrow & H_*^{\infty}(Y,\mathfrak{s}) & \longrightarrow & H_*^{+}(Y,\mathfrak{s})\cdots
\end{array}$$

where the vertical arrows are isomorphisms and both rows are long-exact sequences. All homomorphisms preserve the relative gradings and the respective module structures.

Theorem (Attaching 1-handles)

There exists a commutative diagram

$$\begin{array}{ccccc}
\cdots H_*^-(Y,\mathfrak{s}) & \to & H_*^\infty(Y,\mathfrak{s}) & \to & H_*^+(Y,\mathfrak{s})\cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \widehat{HM}_*(M,\mathfrak{s},c_b)\otimes \hat{\mathrm{V}}^{\otimes g} & \to & \overline{HM}_*(M,\mathfrak{s},c_b)\otimes \hat{\mathrm{V}}^{\otimes g} & \to & \widecheck{HM}_*(M,\mathfrak{s},c_b)\otimes \hat{\mathrm{V}}^{\otimes g}\cdots
\end{array}$$

where the vertical arrows are isomorphisms and both rows are long-exact sequences. All homomorphisms preserve the relative gradings and the respective module structures.

Remark

J. Bloom, T. Mrowka, and P. Ozsváth proved a general connected sum formula that involves completed versions of Seiberg–Witten Floer homology groups.

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