EMBEDDING COMPLEX TORI INTO PROJECTIVE SPACE

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ABSTRACT. We describe the Riemann conditions and the Riemann-Roch theorem on complex tori. As an application, we embed complex tori into projective space to get abelian varieties.

1. INTRODUCTION

Let X be a complex torus. We would like to know when X is an algebraic variety. It can be shown that if X is algebraic, then X is also projective, and then pulling back the hyperplane bundle on projective space gives a very ample line bundle on X. Conversely, if we can embed X as a complex submanifold of projective space, then by Chow's theorem, X will be algebraic. Hence our interest will be holomorphic line bundles on X with lots of holomorphic sections.

The hyperplane bundle $\mathscr{O}_{\mathbf{P}^n}(1)$ on projective space admits a Hermitian metric which pulls back to a Hermitian metric H on $\mathscr{O}_X(1)$. For any choice of H, $\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}H$ is a Kähler form on X, meaning that it is a closed, real-valued, positive definite differential form of type $(1,1)$. Furthermore, this Kähler form is *integral*. This means that its de Rham cohomology class in $H^2(X,\mathbf{C})$ lies in the image of $H^2(X,\mathbf{Z})$. So a necessary condition for X to be a projective variety is that X has an integral Kähler form.

The Kodaira Embedding Theorem states that having an integral Kähler form is also a sufficient condition for being a projective variety. The structure of complex tori allows us to give a precise criterion, called the Riemann relations, for when a complex torus admits an integral Kähler form. This structure also allows us to give a much simpler proof of the Kodaira Embedding Theorem via the Riemann-Roch theorem. We will not complete the proof in this talk, but we will prove the Riemann-Roch theorem for positive line bundles.

2. THE RIEMANN RELATIONS

Write X as the quotient of a vector space V by a lattice Γ. Recall that classes in $H^2(X,\mathbf{Z})$, which are represented by closed integral differential 2-forms on X modulo exact 2-forms, are equivalent to integral alternating bilinear forms on V. So to study integral Kähler forms on X our first step will be to study alternating forms on V.

Proposition 2.1. *Let ω be a non-degenerate alternating bilinear form $V \times V \to \mathbf{C}$ which is integral on a lattice Γ. Then there exist positive integers $d_1,\dots,d_g$ such that $d_1|d_2$, $d_2|d_3$, $\dots, d_{g-1}|d_g$ and a real basis $\gamma_1,\dots,\gamma_{2g}$ of Γ in which the matrix of ω is*

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}. \tag{1}$$

Here Δ is the diagonal matrix with entries $d_1,\dots,d_g$. Furthermore, $d_1,\dots,d_g$ are unique, that is, any other matrix for ω of the form (1) has the same $d_1,\dots,d_g$.

The basis $\gamma_1,\ldots,\gamma_{2g}$ is called a *symplectic basis* for ω. If we were working with real numbers instead of integers, it would be possible to make each d equal one. The product $d_1\cdots d_g$ is called the *Pfaffian* of ω and is denoted by $\mathrm{pf}(\omega)$. Notice that $\mathrm{pf}(\omega)^2=\det\omega$.

Proof. We apply induction on the rank of the vector space. Let d_1 be the smallest positive value of ω on Γ. Say that this is achieved by γ_1 and γ_{g+1}, that is, $\omega(\gamma_1,\gamma_{g+1})=d_1$. For any γ in Γ, d_1 is less than or equal to $\omega(\gamma_1,\gamma)$ and $\omega(\gamma,\gamma_{g+1})$. By the division algorithm and minimality, d_1 must divide these two quantities. Therefore

$$\gamma-\frac{\omega(\gamma_1,\gamma)}{d_1}\gamma_{g+1}-\frac{\omega(\gamma,\gamma_{g+1})}{d_1}\gamma_1$$

lies in Γ. Because ω is alternating, this element is orthogonal to γ_1 and γ_{g+1}, so we get an orthogonal (with respect to ω) decomposition

$$\Gamma=(\mathbf{Z}\gamma_1\oplus\mathbf{Z}\gamma_{g+1})\oplus(\mathbf{Z}\gamma_1\oplus\mathbf{Z}\gamma_{g+1})^{\perp}.$$

Let Γ' be the lattice $(\mathbf{Z}\gamma_1\oplus\mathbf{Z}\gamma_{g+1})^{\perp}$, and let V' be the vector subspace of V spanned by Γ'. By the inductive hypothesis, we can find integers $d_2,\ldots,d_g$ and a basis $\gamma_2,\ldots,\gamma_g,\gamma_{g+2},\ldots,\gamma_{2g}$ of V' which satisfy the theorem for $\omega|_{V'}$. In the basis $\gamma_1,\ldots,\gamma_{2g}$, the matrix of ω has the form (1). The claim that $d_1|d_2$ does not immediately follow from the assertions above: Above we showed only that d_1 divides all the values of $\omega(\gamma_1,-)$ and of $\omega(-,\gamma_{g+1})$. To see that $d_1|d_2$, write $d_2=qd_1+r$ with $0\le r<d_1$ by the division algorithm. By orthogonality, we have

$$\omega(x-q\gamma_1,y+\gamma_{g+1})=r,$$

and because d_1 was chosen minimal, we must have $r=0$.

To see uniqueness, we need a lemma.

Lemma 2.2. *Let M be an integer matrix of size $n\times n$, and let $r\le n$. Then the greatest common divisor of the set of determinants of $r\times r$ minors of M is invariant under the action of* $\mathrm{SL}(n,\mathbf{Z})$.

Proof. First, we reduce to the case when r is 1. To do this, construct a matrix N of size $\binom{n}{r}\times\binom{n}{r}$ as follows: Each row of N corresponds to a choice of r rows from M, and each column to a choice of r columns from M. Together these determine an $r\times r$ submatrix of M, and the corresponding entry of N is the determinant of this $r\times r$ submatrix. To prove the lemma it suffices to show that the greatest common divisor of the entries of N is preserved under the action of $\mathrm{SL}(n,\mathbf{Z})$.

Let d be the greatest common divisor of the entries of N. Choose a matrix S in $\mathrm{SL}(n,\mathbf{Z})$, and let e be the greatest common divisor of the entries of SN. Every entry of SN is a linear combination of entries of N, so $d|e$. By applying the same reasoning to $S^{-1}(SN)=N$ we find that $e|d$, so $d=e$. □

In fact, the lemma holds for matrices of any size over any principal ideal domain. Furthermore, the action of $\mathrm{GL}(n,\mathbf{Z})$ will preserve the greatest common divisor of the set of determinants of $r\times r$ minors of M up to sign. In particular, if M is conjugated by a matrix in $\mathrm{GL}(n,\mathbf{Z})$, then even the sign is preserved.

Now we apply the theorem to the $r\times r$ minors of the matrix of ω in the basis $\gamma_1,\ldots,\gamma_{2g}$. If $r=2s$, then the greatest common divisor is $d_1^2\cdots d_s^2$, and if $r=2s+1$, then the greatest common divisor is $d_1^2\cdots d_s^2d_{s+1}$. Consequently $d_1,d_1^2,d_1^2d_2,d_1^2d_2^2,\ldots,d_1^2\cdots d_g^2$ are all invariant under change of basis. Hence $d_1,\ldots,d_g$ are invariant under change of basis, proving uniqueness. □

Let ω be an integral Kähler form on $X = V/\Gamma$. Find the basis $\gamma_1, \ldots, \gamma_{2g}$ of the previous proposition, and let W be the real vector subspace of V generated by $\gamma_1, \ldots, \gamma_g$. Because ω is positive definite, $\omega(x, ix) > 0$ for any x in W. Therefore, upon inspecting the matrix of ω, we find that ix cannot be in W. Hence $W \cap iW = 0$, so by dimension $V = W \oplus iW$ and $\gamma_1, \ldots, \gamma_g$ is a complex basis for V. Notice that we do not necessarily have $\gamma_{g+j} = i\gamma_j$. Instead, there is some matrix τ relating $i\gamma_1, \ldots, i\gamma_g$ and $\gamma_{g+1}, \ldots, \gamma_{2g}$, and the Riemann conditions are a condition on this matrix:

Theorem 2.3 (Riemann relations)**.** *The complex torus V/Γ admits an integral Kähler form if and only if there exists a complex basis $\mathscr{B} = \{e_1, \ldots, e_g\}$ of V, positive integers $d_1, \ldots, d_g$ such that $d_1 | d_2, \ldots, d_{g-1} | d_g$, and a complex symmetric matrix τ of size $g \times g$ and rank g with positive definite imaginary part such that in the basis $\mathscr{B}$, $\Gamma = \tau\mathbf{Z}^g \oplus \Delta\mathbf{Z}^g$, where Δ is the diagonal matrix with entries $d_1, \ldots, d_g$.*

Proof. We keep the same notation as before. Most of the "only if" direction is clear from our considerations above by setting $e_1 = \gamma_1/d_1, \ldots, e_g = \gamma_g/d_g$. Symmetry of τ will follow from our computations below. To complete the "if" direction, we let $(\Delta\ \tau)$ be the *period matrix* of $\gamma_1, \ldots, \gamma_{2g}$. That is, $(\Delta\ \tau)$ is a $g \times 2g$ matrix whose columns are the vectors $\gamma_1, \ldots, \gamma_{2g}$ (in order). Clearly we have the relations $e_i = \gamma_i/d_i$ for all i. Now we declare $\gamma_1, \ldots, \gamma_{2g}$ to be a symplectic basis and take (1) as the matrix of ω. Clearly ω is alternating, integral, and positive definite, so we need only check that it is of type $(1,1)$. This is true if and only if $\omega(ix, iy) = \omega(x, y)$ for all x and y. Because $e_1, \ldots, e_g$ are a complex basis, it suffices to compare $\omega(e_j, e_k)$ and $\omega(ie_j, ie_k)$ for all j and k.

Split τ into its real part R and its imaginary part S. The matrix $\left(\begin{smallmatrix}\Delta & R\\ 0 & S\end{smallmatrix}\right)$ is the matrix of coefficients of $\gamma_1, \ldots, \gamma_{2g}$ in terms of the real basis $e_1, \ldots, e_g, ie_1, \ldots, ie_g$. By inverting this matrix we get the coefficients of $e_1, \ldots, e_g, ie_1, \ldots, ie_g$ in the basis $\gamma_1, \ldots, \gamma_{2g}$, and we know the matrix of ω in the latter basis. This allows us to evaluate ω on all pairs of vectors in the basis $e_1, \ldots, e_g, ie_1, \ldots, ie_g$:

$$\begin{aligned}
&\left(\begin{pmatrix}\Delta & R\\ 0 & S\end{pmatrix}^{-1}\right)^T \begin{pmatrix}0 & \Delta\\ -\Delta & 0\end{pmatrix}\begin{pmatrix}\Delta & R\\ 0 & S\end{pmatrix}^{-1}\\
&= \left(\begin{pmatrix}\Delta & R\\ 0 & S\end{pmatrix}\begin{pmatrix}0 & \Delta^{-1}\\ -\Delta^{-1} & 0\end{pmatrix}\begin{pmatrix}\Delta^T & 0\\ R^T & S^T\end{pmatrix}\right)^{-1}\\
&= \left(\begin{pmatrix}-R\Delta^{-1} & 1\\ -S\Delta^{-1} & 0\end{pmatrix}\begin{pmatrix}\Delta & 0\\ R^T & S^T\end{pmatrix}\right)^{-1}\\
&= \begin{pmatrix}R^T - R & S^T\\ -S & 0\end{pmatrix}^{-1}\\
&= \begin{pmatrix}0 & -S^{-1}\\ (S^T)^{-1} & (S^T)^{-1}(R^T - R)S^{-1}\end{pmatrix}
\end{aligned}$$

For ω to be type $(1,1)$, therefore, we must have $S^{-1} = (S^T)^{-1}$ and $(S^T)^{-1}(R^T - R)S^{-1} = 0$. Since τ is symmetric, this is true. □

3. The Riemann-Roch Theorem

The Riemann relations, together with the Kodaira Embedding Theorem and Chow's theorem, give a condition for a complex torus to be an abelian variety. Instead of relying on the full strength of the Kodaira Embedding Theorem, we can give a more direct proof using the structure of complex tori. The first step in this proof is a case of the Riemann-Roch theorem for complex tori.

Theorem 3.1 (Riemann-Roch)**.** *Let X be a complex torus, and let L be a line bundle on X with positive definite first Chern class. Then*

$$\dim H^0(X,L) = \mathrm{pf}(c_1(L)) > 0.$$

The assumption that L has positive definite first Chern class is usually stated by saying that L is *positive*. In general, the Kodaira Embedding Theorem says that this is equivalent to L being ample. Later in this series of lectures we will see this directly for complex tori (but not in this talk). The Riemann-Roch theorem alone is not sufficient to prove ampleness because it does not guarantee anything about the sections; for example, the complete linear system on L could have a base locus.

Proof. The essential idea is to determine whether a function is a theta function in terms of conditions on its Fourier coefficients. By the Appell-Humbert theorem, we may assume that L is of type (H,α) for some Hermitian form H and some $\alpha : \Gamma \to U(1)$ such that $\alpha(\gamma_1+\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)(-1)^{\omega(\gamma_1,\gamma_2)}$. A section of such a line bundle is a theta function which satisfies, for any z in V and γ in Γ,

$$\theta(z+\gamma) = \alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\theta(z)$$

To determine whether a function on V is a theta function, it suffices to check this relation whenever γ is one of $\gamma_1,\dots,\gamma_{2g}$. To do this we will use a different normalization of the theta function called the *classical normalization*. The classical normalization will be periodic, so classical theta functions have Fourier series.

Restrict α to the subgroup Γ' generated by $\gamma_1,\dots,\gamma_g$. Because $\gamma_1,\dots,\gamma_g$ is the first half of a symplectic basis, α becomes a group homomorphism after being so restricted. Therefore, there is a $\mathbf{C}$-linear form ℓ on V such that $\alpha(\gamma) = e^{2\pi i\ell(\gamma)}$ for all γ in Γ'. Define B to be the $\mathbf{C}$-linear bilinear form on V gotten by extending scalars on the $\mathbf{R}$-linear form $H|_W$, where W is the real vector space generated by $\gamma_1,\dots,\gamma_g$. B is symmetric. We let

$$\tilde{\theta}(z) = e^{-\frac{\pi}{2}B(z,z)-2\pi i\ell(z)}\theta(z).$$

This is the *classical theta function* of type (H,α). The transformation rule for this function is

$$\begin{aligned}
\tilde{\theta}(z+\gamma) &= e^{-\frac{\pi}{2}B(z+\gamma,z+\gamma)-2\pi i\ell(z+\gamma)}\theta(z+\gamma)\\
&= e^{-\frac{\pi}{2}B(z+\gamma,z+\gamma)-2\pi i\ell(z+\gamma)}\alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\theta(z)\\
&= e^{-\frac{\pi}{2}(B(z,z)+B(z,\gamma)+B(\gamma,z)+B(\gamma,\gamma))-2\pi i(\ell(z)+\ell(\gamma))}\alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\theta(z)\\
&= e^{-\frac{\pi}{2}(B(z,\gamma)+B(\gamma,z)+B(\gamma,\gamma))-2\pi i\ell(\gamma)}\alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\tilde{\theta}(z)\\
&= \alpha(\gamma)e^{-2\pi i\ell(\gamma)}e^{\pi(H-B)(\gamma,z)+\frac{\pi}{2}(H-B)(\gamma,\gamma)}\tilde{\theta}(z)
\end{aligned}$$

If we substitute one of $\gamma_1,\dots,\gamma_g$ for γ in this equation, then, because $H = B$ on W and because $\alpha(\gamma) = e^{2\pi i\ell(\gamma)}$ on Γ', we find that $\tilde{\theta}(z+\gamma) = \tilde{\theta}(z)$. So $\tilde{\theta}$ is periodic in the discrete subgroup generated by $\gamma_1,\dots,\gamma_g$, and hence it has a Fourier series expansion in that complex basis:

$$\tilde{\theta}(\sum_{g=1}^{k} z_k\gamma_k) = \sum_{m\in\mathbf{Z}^g} c(m)e^{2\pi i\sum_k m_k z_k}.$$

For $\tilde{\theta}$ to be a theta function, it must also satisfy a transformation rule for the other generators $\gamma_{g+1},\dots,\gamma_{2g}$ of Γ. For these lattice elements, there are non-zero constants $b_1,\dots,b_g$ such that

$$\tilde{\theta}(z+\gamma_{g+j}) = b_j e^{\pi(H-B)(\gamma_{g+j},z)}\tilde{\theta}(z). \tag{2}$$

Before we expand this equation in Fourier series, we'll determine the exponent on the right-hand side. Notice that for any $x\in V$ and $y\in W$, we have

$$(H-B)(x,y) = (\overline{H}-B)(y,x) = (\overline{H}-H)(y,x) = -2i\omega(y,x) = 2i\omega(x,y),$$

hence

$$(H-B)(\gamma_{g+j},z) = \sum_k z_k(H-B)(\gamma_{g+j},\gamma_k) = -2id_jz_j$$

Using the matrix τ that appeared in the Riemann relations, we can write

$$\gamma_{g+j} = \sum_k \tau_{kj}e_k = \sum_k (\tau_{kj}/d_k)\gamma_k.$$

Substituting these relations into (2) gives

$$\sum_{m\in\mathbf{Z}^g} c(m)e^{2\pi i\left(\sum_k m_kz_k+\sum_k m_k\frac{\tau_{kj}}{d_k}\right)} = b_j\sum_{m\in\mathbf{Z}^g} c(m)e^{2\pi i\left(\sum_k m_kz_k - d_jz_k\right)}.$$

So we want, for all $m\in\mathbf{Z}^g$,

$$c(m)e^{2\pi i\sum_k m_k\frac{\tau_{kj}}{d_k}} = b_kc(m+d_j\varepsilon_j), \tag{3}$$

where ε_j is the jth standard basis vector. In particular, when $\tilde{\theta}$ is a theta function we can determine all the possible values of $c(m)$ in terms of the $c(m)$ when $0\le m_j<d_j$ for all j. Consequently,

$$\dim H^0(X,L)\le d_1\cdots d_g = \mathrm{pf}(\omega).$$

Conversely, we may choose $c(m)$ arbitrarily for $0\le m_j<d_j$ and impose the relation (3). Then it suffices to prove that the resulting Fourier series converges. It is easy to check that $c(m)$ grows like

$$|e^{2\pi i\sum_{j,k}\frac{m_j}{d_j}\frac{m_k}{d_k}\tau_{jk}}|$$

Since the imaginary part of τ is positive definite, the Fourier series converges. □