TRIANGULATIONS OF ALGEBRAIC SETS

KYLE HOFMANN

ABSTRACT. Following an exposition of Hironaka, we show that, locally in the Euclidean topology, every real or complex algebraic variety has a triangulation, that is, it is homeomorphic to a simplicial complex.

1. PIECEWISE LINEAR TOPOLOGY

Recall that a *simplicial complex* is a topological space formed by gluing together simplices. We will consider only locally finite simplicial complices, that is, around each point there is a neighborhood which meets only finitely many simplices. A function $f : K_1 \to K_2$ between two simplicial complices is called *piecewise linear* or *PL* if, after subdividing K_1 and K_2, f sends simplices to simplices. A *triangulation* of a topological space X is a homeomorphism $f : K \to X$ from a simplicial complex to X. Two triangulations $f_1 : K_1 \to X$ and $f_2 : K_2 \to X$ are *equivalent* if $f_2^{-1} f_1$ (equivalently, $f_1^{-1} f_2$) is PL. An equivalence class of triangulations on X is called a *PL structure* on X. Having a PL structure allows the use of standard tools from algebraic topology such as Borel-Moore homology.

Algebraic varieties have underlying topological spaces, and the best that one could hope for is that, over $\mathbf{R}$ or $\mathbf{C}$, the Euclidean topology on an algebraic variety carries a functorial PL structure. Denoting by $X(k)$ the set of all k-points together with its natural Euclidean topology, we would like:

Hope 1.1. *Let k be $\mathbf{R}$ or $\mathbf{C}$. Let Sch_k be the category of finite-type k-schemes, and let PL be the category of topological spaces with PL structure and PL functions. Then there exists a functor $\tau : Sch_k \to PL$ such that*

(1) The underlying topological space of $\tau(X)$ is $X(k)$ for any X.
(2) For any morphism $f : X \to Y$, $\tau(f)$ is the pushforward of k-valued points along f.

While this is widely believed to be true, a proof has never appeared in the literature. What is known for certain is that a variety can be triangulated locally in the Euclidean topology. That is, around any point $x \in X$, there is a small Euclidean neighborhood U which can be triangulated. This is the result we will present. We will follow the exposition of Hironaka [2], which is based on the Łojasiewicz's proof of triangulation for semi-analytic sets [4]. The difficulty in globalizing this result consists of trying to make sure that two triangulations are compatible on overlaps. We will have more to say about this later.

This is not the only sort of topological structure one could put on a variety. If X is a smooth affine variety, then one can make it into a CW complex using Morse theory. If X is smooth and compact, then it embeds as a smooth compact manifold into a Euclidean space, and again Morse theory applies. But if X is singular, then Morse theory does not work. Even if X is smooth, if $Y \subseteq X$ is a smooth subvariety, Morse theory does not guarantee that one can associate to Y a cohomology class in H^*X, whereas if X and Y are triangulated appropriately, this is clear.

2. Semi-algebraic sets

The proof we will give is inductive: It proceeds by projecting onto a hyperplane in a good way and lifting a triangulation from the projection. One immediate problem is that the sets we are interested in are algebraic, that is, cut out by equations, but their projections are not. For example, project the circle $\{x^2+y^2-1=0\}$ in $\mathbf{R}^2$ onto the x-axis. The resulting set is $\{-1 \le x \le 1\}$, which is clearly not algebraic. So we must work with a slightly more general notion, called a semi-algebraic set.

Definition 2.1. $X \subseteq \mathbf{R}^n$ is *semi-algebraic* if it is generated by finite unions, intersections, and complements of sets of the form

$$\{x \mid f(x) \ge 0\} \tag{1}$$

Notice that

$$\{f(x) > 0\} = \mathbf{R}^n \setminus \{-f(x) \ge 0\},$$
$$\{f(x) = 0\} = \{f(x) \ge 0\} \cap \{-f(x) \ge 0\}.$$

Furthermore, X is semi-algebraic if and only if there exist polynomials f_{ij} and g_{ij} such that

$$X = \bigcup_i \{x \mid \forall j,\ f_{ij}(x) > 0 \text{ and } g_{ij}(x) = 0\}. \tag{2}$$

By replacing all the strict inequalities in (2) by weak inequalities, one gets $\overline{X}$, so $\overline{X}$ is semi-algebraic. It follows that X°, the interior of X, is also semi-algebraic.

Definition 2.2. Let $A \subseteq \mathbf{R}^n$ and $B \subseteq \mathbf{R}^m$ be semi-algebraic. A continuous function $f : A \to B$ is *semi-algebraic* if its graph is semi-algebraic in $\mathbf{R}^n \times \mathbf{R}^m$.

Compositions and inverses of semi-algebraic maps are semi-algebraic. The most basic fact non-trivial fact about semi-algebraic sets is that all polynomial maps are semi-algebraic:

Theorem 2.3 (Tarski [6], Seidenberg [5]). *If $X \subseteq \mathbf{R}^n$ is semi-algebraic and if $f : \mathbf{R}^n \to \mathbf{R}^m$ is a polynomial map, then $f(X)$ is semi-algebraic.*

This theorem is effective, so in principle it is possible to find equations and inequalities for $f(X)$ from those defining X. It follows from this theorem that images of semi-algebraic sets under semi-algebraic maps are semi-algebraic.

Our main tool for controlling the behavior of a semi-algebraic set will be stratifications. For our purposes, the following type of stratification will be the most useful.

Definition 2.4. By a *stratification* of a subset $A \subseteq \mathbf{R}^n$ we will mean a collection of sets $\Gamma_\mu \subseteq \mathbf{R}^n$ such that:

(1) A is the disjoint union of all the Γ_μ.
(2) $\{\Gamma_\mu\}$ is locally finite about each point of $\mathbf{R}^n$. (In particular, it is finite about points of the boundary of A, even if those points are not in A itself.)
(3) Each Γ_μ is a smooth connected real-analytic locally closed submanifold of $\mathbf{R}^n$.
(4) If $\overline{\Gamma}_\mu \cap \Gamma_\nu \ne \emptyset$, then $\overline{\Gamma}_\mu \supset \Gamma_\nu$.

A stratification is *finite* if there are only finitely many μ. It is *semi-algebraic* if each Γ_μ is semi-algebraic.

Finally, we make a convention: *All simplices are open*. That is, the n-simplex in $\mathbf{R}^{n+1}$ is the set $\{(x_0,\dots,x_n) \mid x_0,\dots,x_n > 0 \text{ and } x_0+\cdots+x_n = 1\}$, not the similarly defined set with $x_0,\dots,x_n \geq 0$.

Now we can state the main theorem on triangulation:

Theorem 2.5. *Let $\{X_\alpha\}$ be a finite collection of bounded semi-algebraic sets in $\mathbf{R}^n$. Then there exists a simplicial decomposition $\mathbf{R}^n = \bigcup_a \Delta_a$ and a semi-algebraic automorphism κ of $\mathbf{R}^n$ such that:*

1. *The collection of all $\kappa(\Delta_a)$ is a stratification of $\mathbf{R}^n$ in the sense above.*
2. *Each X_α is a finite union of some subcollection of the $\kappa(\Delta_a)$.*
3. *For all a, $\kappa : \Delta_a \to \kappa(\Delta_a)$ is a real-analytic isomorphism.*
4. *Outside of a compact subset G, κ is the identity.*

It follows that κ determines a homeomorphism from a simplicial complex to $\bigcup_\alpha X_\alpha$, thereby triangulating the latter.

3. Proof of the triangulation theorem

The theorem is obvious when $n = 1$. The rest of the proof is an induction on n, and it has the following outline:

I. Reduce to the case where every X_α is closed in $\mathbf{R}^n$.
II. Reduce to the case where every X_α has empty interior.
III. There exists a projection $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ such that for all $y \in \mathbf{R}^{n-1}$ and all α, $\pi^{-1}(y) \cap X_\alpha$ is a discrete set. After a change of coordinates, we may assume that $\pi(x_1,\dots,x_n) = (x_1,\dots,x_{n-1})$.
IV. Let $D = \pi(X)^\circ$. Reduce to the case where $X \cap \pi^{-1}(D)$ is dense in X, $\pi|_{\pi^{-1}(D)}$ is open, and $X \setminus \pi^{-1}(D) \subseteq \{x \mid x_n = 0\}$.
   A. There exists a finite semi-algebraic stratification $\{\Gamma_\mu\}$ of $\{X_\alpha\}$.
   B. Reduce to the case where $\pi_\mu \stackrel{\text{def}}{=} \pi|_{\Gamma_\mu}$ is a real-analytic immersion for all μ.
   C. By induction, choose a simplicial decomposition $\mathbf{R}^{n-1} = \bigcup_a \Delta_a$ and a semi-algebraic automorphism κ of $\mathbf{R}^{n-1}$ as in the theorem. Using these, reduce to the case where π_μ is a real-analytic isomorphism for all μ.
V. Reduce to the case where κ is the identity outside $\mathbf{R}^{n-1} \setminus D$.
VI. Reduce to the case where, for all μ and ν, if $\overline{\Gamma}_\mu \cap \overline{\Gamma}_\nu \neq \emptyset$, then there exists c such that $\pi(\overline{\Gamma}_\mu \cap \overline{\Gamma}_\nu) = \kappa(\overline{\Delta}_c)$.
VII. Extend κ to a semi-algebraic automorphism $\hat\kappa$ of $\mathbf{R}^n$ such that
   1. For all μ, $\hat\kappa^{-1}(\Gamma_\mu)$ is a simplex in $\mathbf{R}^n$.
   2. For all μ, $\hat\kappa|_{\hat\kappa^{-1}(\Gamma_\mu)}$ is a real-analytic isomorphism.
   3. $\hat\kappa$ is the identity outside of a compact subset $G \subseteq \mathbf{R}^n$.
VIII. Choose a compatible simplicial decomposition $\mathbf{R}^n = \bigcup_d \Delta_d$.

Proof of I. Let $X_{\alpha,0} = X$. Recursively define $X_{\alpha,i+1} = \overline{X}_{\alpha,i} \setminus X_{\alpha,i}$. At every step this lowers the dimension by at least 1, that is, $\dim X_{\alpha,i+1} < \dim X_{\alpha,i}$. Since

$$X_\alpha = \overline{X}_{\alpha,0} \setminus X_{\alpha,1} = \overline{X}_{\alpha,0} \setminus (\overline{X}_{\alpha,1} \setminus X_{\alpha,2}) = \cdots = \overline{X}_{\alpha,0} \setminus (\overline{X}_{\alpha,1} \setminus (\overline{X}_{\alpha,2} \setminus \cdots))$$

we can replace the set of all X_α by the set of all $\overline{X}_{\alpha,i}$. □

Proof of II. X_α° is a union of connected components of $\mathbf{R}^n \setminus \partial X_\alpha$. Simplices are connected, so if $\mathbf{R}^n = \bigcup_{a \in A} \Delta_a$ and, for some $A' \subseteq A$, $\partial X_\alpha = \bigcup_{a \in A'} \Delta_a$, then X_α° is the union of some of the simplices Δ_a. Consequently we may replace each X_α by ∂X_α. Note that $\dim \partial X_\alpha \leq n-1$, so each ∂X_α has empty interior. □

Proof of III. By the following lemma, most projections satisfy the condition of III.

Lemma 3.1 (Koopman-Brown [3])**.** *Let Z be a locally closed real-analytic nowhere dense subset of $\mathbf{R}^n$. Then*

$$B(Z) = \{v \in \mathbf{RP}^{n-1} \mid Z \textit{ contains an open subset of a line in the direction of } v\}$$

is meager (a countable union of nowhere dense subsets).

The statement is local, since $\mathbf{R}^n$ is second countable. In the local case it can be turned into a statement about partial derivatives, and the essential observation is that vanshing loci of (not identically zero) analytic functions are nowhere dense.

Because each X_α has empty interior, they are all nowhere dense, so by the lemma we may choose a projection. Up to a change of coordinates, it is projection onto the first $n-1$ coordinates. □

Proof of IV.A. Write the semi-algebraic sets as in (2), and let $\mathscr{E}$ be the set of all f_{ij} and g_{ij} occuring in those equations. Consider the set of all functions $\mu : \mathscr{E} \to \{<, =, >\}$. Each μ determines a subset Γ_μ of $\mathbf{R}^n$:

$$\Gamma_\mu = \{(x_1, \ldots, x_n) \mid h \;\; \mu(h) \;\; 0 \text{ for all } h \in \mathscr{E}\}.$$

Every X_α is a disjoint union of some of the Γ_μ, and $\mathbf{R}^n$ is the disjoint union of all of the Γ_μ. If one of the Γ_μ is singular, then we decompose it further as the union of its smooth locus and its singular locus and repeat until each Γ_μ is smooth. Similarly if one of the Γ_μ is disconnected. This determines a stratification. □

Proof of IV.B. Use IV.A to construct a stratification $\{\Gamma_\mu\}$. Recall that $\pi_\mu = \pi|_{\Gamma_\mu}$. Let $S_\mu = \{x \in \Gamma_\mu \mid \operatorname{rank}(d\pi_\mu)_x < \dim \Gamma_\mu\}$. S_μ is semi-algebraic. Applying IV.A again, we can stratify $\{X_\alpha\} \cup \{S_\mu\}$ to get $\{\Gamma'_\nu\}$. $\{\Gamma'_\nu\}$ refines the stratification $\{\Gamma_\mu\}$. We define S'_ν analogously to S_μ: $S'_\nu = \{x \in \Gamma'_\nu \mid \operatorname{rank}(d\pi_\nu)_x < \dim \Gamma'_\nu\}$. For all nonempty S'_ν, there is a μ such that $S'_\nu \subset \Gamma'_\nu \subseteq S_\mu$, so $\dim S'_\nu < \dim \Gamma'_\nu \le \dim S_\mu$. Applying IV.A recursively, eventually all S'_ν will be empty, so every π_μ will be an immersion. □

Proof of IV.C. Every π_μ is a 1-1 immersion, so it is locally an isomorphism onto its image. That is, around every point x in $\pi_\mu(\Gamma_\mu)$, there is a small open neighborhood U_x such that $\pi_\mu^{-1}(U_x) \cap \pi_\mu(\Gamma_\mu)$ is a disjoint union of copies of $U_x \cap \pi_\mu(\Gamma_\mu)$. Since U_x is open, it contains a ball. Therefore it contains a simplex, and simplices are semi-algebraic. Hence we can cover $\pi_\mu(\Gamma_\mu)$ by open semi-algebraic sets over which π_μ is locally an isomorphism. Since each Γ_μ is bounded, this cover can be chosen to be finite.

For every μ, choose such a cover C_μ of $\pi_\mu(\Gamma_\mu)$. Apply the induction hypothesis to the collection $\{\pi(\Gamma_\mu)\} \cup \bigcup_\mu C_\mu$. This yields a simplicial decomposition $\mathbf{R}^{n-1} = \bigcup_a \Delta_a$ and a semi-algebraic automorphism κ of $\mathbf{R}^{n-1}$ which has the properties listed in the statement of the theorem with respect to the covers C_μ and all the $\pi_\mu(\Gamma_\mu)$. Construct a new stratification Γ'_λ whose strata are the connected components of $\Gamma_\mu \cap \pi^{-1}(\kappa(\Delta_a))$ for all μ and a. Γ'_λ refines Γ_μ, so if we set $\pi'_\lambda = \pi|_{\Gamma'_\lambda}$, then every π'_λ is a finite-to-one real-analytic immersion. By our choice of C_μ, every π'_λ is one-to-one, so a real-analytic isomorphism.

At this point, we change notation: We forget about our original Γ_μ and replace it by Γ'_λ. □

Proof of IV. Extend κ to $\tilde{\kappa}$ on $\mathbf{R}^n$ by letting $\tilde{\kappa}(y,x_n)=(\kappa(y),x_n)$. Choose a large ball B of radius N in $\mathbf{R}^{n-1}$ which contains all $\overline{\Gamma}_\mu$ and such that $\kappa|_{\mathbf{R}^{n-1}\setminus B}$ is the identity, and let S be the boundary of B. For all μ, we will construct a *semi-algebraic tent* from $\tilde{\kappa}^{-1}(\Gamma_\mu)$ to S. This is a semi-algebraic set Z_μ in $\mathbf{R}^n$ which contains $\tilde{\kappa}^{-1}(\Gamma_\mu)$ and S and such that π induces a homeomorphism from Z_μ to $\overline{B}$. In pictures, it looks like a big tent anchored to $\mathbf{R}^{n-1}$ along S and draped over $\tilde{\kappa}^{-1}(\Gamma_\mu)$.

To construct it, find the a such that $\pi(\tilde{\kappa}^{-1}(\Gamma_\mu))=\Delta_a$. Choose a point ξ of Δ_a such that the translation $\Delta_a+\xi$ lies in B. Roughly, ξ will be the center of the tent. Let L be the linear span in $\mathbf{R}^{n-1}$ of Δ_a, and let $s:\mathbf{R}^{n-1}\to L$ be the orthogonal projection. The tent will consist of line segments from points x of S to points $\phi^*(x)$ of $\tilde{\kappa}^{-1}(\Gamma_\mu)$). The x_n coordinate of $\phi^*(x)$ is determined uniquely because π is an isomorphism on $\tilde{\kappa}^{-1}(\Gamma_\mu)$, so to determine the line segment from x it suffices to determine the point $\phi(x)=\pi(\phi^*(x))$ in Δ_a.

Choose $x\in S$. $s(x+\xi)$ is in L but possibly not in Δ_a. Let $H(x)$ be the ray from ξ in the direction of $s(x+\xi)$, and let $h(x)$ be the length of $H(x)\cap\Delta_a$. Define

$$\phi(x)=\xi+\frac{h(x)}{N}(s(x+\xi)-\xi).$$

Take Z_μ to be the union of all the closed line segments connecting x and $\phi^*(x)$ for all $x\in S$. It's clear that Z_μ contains Y and S and induces the desired homeomorphism.

Now we let $\{X'_\beta\}$ be the collection of all X_α and all Z_μ. If $X'=\bigcup_\beta X'_\beta$ and $D'=\pi(X')^\circ$, then X' and D' satisfy the conditions of IV. Since $\{X'_\beta\}$ contains $\{X_\alpha\}$, we replace $\{X_\alpha\}$ by $\{X'_\beta\}$.

Having made this replacement, we lose $\{\Gamma_\mu\}$, κ, and the simplicial decomposition of $\mathbf{R}^{n-1}$. Since we will need them to continue, at this point we rerun steps IV.A–C. As before we obtain a stratification $\{\Gamma_\mu\}$, a simplicial decomposition $\mathbf{R}^{n-1}=\bigcup_a\Delta_a$, and a semi-algebraic automorphism κ of $\mathbf{R}^{n-1}$, and these have the same properties as before, but for the new collection of X_α. □

Proof of V. View each $\kappa(\Delta_a)$ as a subset of $\mathbf{R}^n$ using the inclusion $\mathbf{R}^{n-1}\subset\mathbf{R}^n$. κ is the identity outside of a compact subset of $\mathbf{R}^{n-1}$, so outside the union of finitely many simplices, κ is the identity. For each of these finitely many simplices Δ_a, add $\overline{\kappa(\Delta_a)}$ to the collection of X_α and add $\kappa(\Delta_a)$ to the collection of Γ_μ. For any simplex Δ_b on the boundary of Δ_a, also add $\kappa(\Delta_b)$ to the collection of Γ_μ. This does not disturb any of the properties attained in the previous steps, but it enlarges D so that κ is the identity outside D. □

Proof of VI. Suppose that $\overline{\Gamma}_\mu\cap\overline{\Gamma}_\nu\neq\emptyset$. $\kappa^{-1}(\pi(\overline{\Gamma}_\mu))$ and $\kappa^{-1}(\pi(\overline{\Gamma}_\nu))$ are the closures of simplices. By subdividing these simplices, we may assume that $\pi(\overline{\Gamma}_\mu)\cap\pi(\overline{\Gamma}_\nu)=\kappa(\overline{\Delta}_c)$ for some c. The refined set of simplices is still finite. □

Proof of VII. We will construct the automorphism $\hat{\kappa}$ piecewise. We choose G as follows: Pick a large real number M such that all the Γ_μ are contained in

$$Q_0=\overline{D}\times(-M,M).$$

Let

$$P_+=(0,\dots,0,M+1),$$
$$P_-=(0,\dots,0,-M-1),$$

and take cones over these points as follows:

$$Q_+ = \text{The cone from } P_+ \text{ to } \overline{D} \times \{M\},$$
$$Q_- = \text{The cone from } P_- \text{ to } \overline{D} \times \{-M\}.$$

We set

$$G = Q_- \cup Q_0 \cup Q_+.$$

It is easy to define $\hat{\kappa}$ outside G:

$$\hat{\kappa}|_{\mathbf{R}^n \setminus G} = \text{id}.$$

Notice that $\tilde{\kappa}$ is the identity in that region. To construct $\hat{\kappa}$ elsewhere, we will adjust $\tilde{\kappa}$. First we will adjust it in Q_0 using a *semi-algebraic shift*:

For each μ, call a point of $\tilde{\kappa}^{-1}(\overline{\Gamma}_\mu)$ a *vertex* if it lies over a vertex of the simplex $\pi(\tilde{\kappa}^{-1}(\overline{\Gamma}_\mu)) = \kappa^{-1}(\pi(\overline{\Gamma}_\mu))$. Let E_μ be the vertex spanned by the simplices of $\tilde{\kappa}^{-1}(\overline{\Gamma}_\mu)$. Fix a, and consider the collection of all Γ_μ such that $\tilde{\kappa}^{-1}(\Gamma_\mu)$ lies over the simplex Δ_a. We know that each Γ_μ maps isomorphically down onto Δ_a and that their union is a covering map. For each $y \in \Delta_a$, we have two collections of points above y: The points lying in the Γ_μ, and those lying in the E_μ. These points are ordered by their last coordinate, so if we let $\Gamma_\mu \cap \pi^{-1}(y) = \gamma_\mu(y)$ and $E_\mu \cap \pi^{-1}(y) = e_\mu(y)$, then we can order $\{\gamma_\mu(y)\}$ and $\{e_\mu(y)\}$ by last coordinate. That is, if we use a subscript n to denote the last coordinate, we can index the possible μ by integers such that

$$-M < \gamma_1(y)_n < \cdots < \gamma_r(y)_n < M,$$
$$-M < e_1(y)_n < \cdots < e_r(y)_n < M.$$

We set $\gamma_0(y) = e_0(y) = (y, -M)$ and $\gamma_{r+1}(y) = e_{r+1}(y) = (y, M)$. For every $x \in \{y\} \times [-M, M]$, find i such that

$$\gamma_i(y)_n \le x_n \le \gamma_{i+1}(y)_n.$$

Then find $\alpha(x)$ such that

$$x_n = \alpha(x)\gamma_i(y)_n + (1-\alpha(x))\gamma_{i+1}(y)_n.$$

Define

$$h_a(x) = \alpha(x)e_i(y) + (1-\alpha(x))e_{i+1}(y).$$

h_a is the desired semi-algebraic shift. It is an automorphism of $\overline{Q}_0 \cap \pi^{-1}\kappa(\Delta_a)$. If $\Delta_b \subset \overline{\Delta}_a$, then, by step IV, if Γ_μ lies over Δ_b, then there is ν such that Γ_ν lies over Δ_a and $\Gamma_\mu \subset \overline{\Gamma}_\nu$. It follows that h_a and h_b together define a continuous automorphism of $\overline{Q}_0 \cap \pi^{-1}\kappa(\Delta_a \cup \Delta_b)$. So the collection of all h_a determine a semi-algebraic automorphism h of $\overline{Q}_0$. It also follows that h is the identity on ∂Q_0. Since κ is the identity outside D, $h\tilde{\kappa}$ is the identity on $Q_0 \setminus \pi^{-1}(D)$. This means that $h\tilde{\kappa}$ glues to the previously defined part of $\hat{\kappa}$, so we define

$$\hat{\kappa}|_{Q_0} = h\tilde{\kappa}|_{Q_0}.$$

To define $\hat{\kappa}$ on Q_+, take the suspension of $h\tilde{\kappa}$ from $\overline{D} \times \{M\}$ to P_+ and call it g_+. Similarly, suspect $h\tilde{\kappa}$ from $\overline{D} \times \{-M\}$ to P_- and call the result g_-. Because $h\tilde{\kappa}$ is the identity on $\partial D \times \{\pm M\}$, $g_\pm$ glues to the rest of $\hat{\kappa}$, so we define

$$\hat{\kappa}|_{Q_\pm} = g_\pm.$$

$\hat{\kappa}$ is a semi-algebraic automorphism, and it induces real-analytic isomorphisms as follows:

(1) $\hat{\kappa}^{-1}(\Gamma_\mu) \to \Gamma_\mu$ for all μ.
(2) $Q_0 \setminus \bigcup_\mu \hat{\kappa}^{-1}(\Gamma_\mu) \to Q_0 \setminus \bigcup_\mu \Gamma_\mu$ for all μ.

(3) (The cone from $\Delta_a \times \{M\}$ to P_+) → (The cone from $\kappa(\Delta_a) \times \{M\}$ to P_+).
(4) (The cone from $\Delta_a \times \{-M\}$ to P_-) → (The cone from $\kappa(\Delta_a) \times \{M\}$ to P_-).

In particular, it satisfies the conditions of VII. □

Proof of VIII. Each of the sets on the left-hand side of the previous list is piecewise linear. Consequently it is clear that there exists a simplicial decomposition of $\mathbf{R}^n$ compatible with all of these sets. □

4. Globalizing

As we indicated at the beginning, the main problem in globalizing is that two triangulations on two open sets need not be compatible. This is a real problem: There are known examples of topological manifolds having distinct PL structures. What one would like to do is as follows:

(1) Reduce to the affine case. Cover the variety by open affines. Assume that the variety can be triangulated on these open sets, and show that these triangulations are compatible, so that they glue to give a global triangulation.
(2) Reduce to the real case. Once one is in the affine case, one can apply Weil restriction of scalars; all that means in this case is using the identification $\mathbf{C}^n = \mathbf{R}^{2n}$ and taking real and imaginary parts of everything.
(3) Reduce to the bounded case. Cover Euclidean space by a locally finite collection of balls. The intersection of a variety with each ball satisfies the hypotheses of the theorem, and one would like to check that if two balls meet, then they induce equivalent triangulations on their intersection.

The first and third steps are the difficult ones. It is probably possible to bypass the third step, because Hironaka also proves that one can triangulate possibly unbounded sub-analytic sets. (See [2] for details and [1] for more about sub-analytic sets.) But it is not possible to bypass the first step.

Hironaka, in the introduction to his paper, claimed that his proof gave a "canonical (up to isomorphisms) PL-structure to each algebraic set." His justification for this seems to be Remark 2.4, where he says,

> Let X be any bounded semi-algebraic set in $\mathbf{R}^n$. Pick any two semi-algebraic triangulations for the same X, say $(\kappa_i, \mathbf{R}^n = \bigcup_a \Delta_{i,a})$, $i = 1,2$. Then let $\{X_\alpha\}$ be the collection of all those $\kappa_i(\Delta_{ia})$ [sic] which are contained in X. It is a finite system of bounded semi-algebraic sets in $\mathbf{R}^n$. Hence there exists a semi-algebraic triangulation for $\{X_\alpha\}$, say $(\kappa_3, \mathbf{R}^n = \bigcup_d \Delta_{3,d})$. Note that for each $i = 1,2$, and for each $\Delta_{i,a}$, $\kappa_i(\Delta_{i,a})$ is a finite union of some of the $\kappa_3(\Delta_{3,d})$. This shows that PL-structures induced on X by the first two triangulations are isomorphic to each other. In this sense, X carries a canonical PL-structure (up to isomorphisms).

While this seems plausible, Hironaka says no more. We leave further investigation to the reader.

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