Systemic Risk in Large Financial Networks

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Outline

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2 Dynamic Default Timing Model
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4 Numerical Solution-Law of Large Numbers
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Part I

Motivation, Objectives & Main Difficulties
Systemic risk:

The likelihood of failure of a lot of components in complex networks.

- In our complex financial world there are many sources of risk, many interactions and correlation between assets.

- Important to understand the behavior of large portfolios and to model correlated default.

- We want to understand the behavior of typically and atypically large default clusters.

- Diversification may lead to robustness, but linkages (correlations) may also provide pathways for contagion.
Motivation, Objectives & Main Difficulties

Goal...

- Development of **mathematical** and **computational** tools for the measurement and prediction of systemic risk in high-dimensional financial networks. Examples:
  1. approximations of distributions of losses from defaults (e.g., large scale behavior, fluctuation theory)
  2. approximations of portfolio risk measures

- Insights into the behavior of systemic risk as a function of its characteristics and their interaction.
Goal...

- Development of mathematical and computational tools for the measurement and prediction of systemic risk in high-dimensional financial networks. Examples:
  1. approximations of distributions of losses from defaults (e.g., large scale behavior, fluctuation theory)
  2. approximations of portfolio risk measures

- Insights into the behavior of systemic risk as a function of its characteristics and their interaction.

- More accurate risk management.
- May inform the design of regulatory policy.
Motivation, Objectives & Main Difficulties

Goal...

- Interest is in large pool of credit assets.

- Model effects of contagion and exposure to exogenous risk.

- Build a well motivated stable complex system and study typical behavior and rare events.

- Find methods for coarse-graining the behavior of the portfolio.
Motivation, Objectives & Main Difficulties

Goal...

Let's consider a large $N \geq 1$ pool of credit assets. Default times are $\{\tau_n\}_{1 \leq n \leq N}$. We are interested in the empirical default rate

$$L^N_t = \frac{1}{N} \sum_{n=1}^{N} \chi\{\tau_n \leq t\}$$

(1)

i.e the number of assets which have defaulted by time $t$. 

What is typical behavior for large $N$? What is a typical loss rate? If enough diversification, we can hope that $L^N_t \approx L(t)$ for large $N$.

How does atypical behavior occur? What is most likely way that the pool suffers an atypically large default rate? If $\ell > L(t)$, what is the most likely way that $\{L^N_t \geq \ell\}$ occurs?
Let’s consider a large $N \geq 1$ pool of credit assets. Default times are $\{\tau_n\}_{1 \leq n \leq N}$. We are interested in the empirical default rate

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i.e the number of assets which have defaulted by time $t$.

1. What is *typical* behavior for large $N$? What is a *typical* loss rate? If enough diversification, we can hope that $L^N_t \approx L(t)$ for large $N$.

2. How does *atypical* behavior occur? What is most likely way that the pool suffers an *atypically* large default rate? If $\ell > L(t)$, what is the most likely way that $\{L^N_t \geq \ell\}$ occurs?
Empirical Motivation

- Variations in **systematic risk factor** and **contagious impact** of defaults on the health of other firms $\Rightarrow$ reasons for a large amount of default clustering (AGS10, DDKS07)

- Constituent intensity depends on the path of the default rate. The impact of a default on the dynamics of surviving firms is **transient (recovery effect)** (AGS10).

- Names in the pool are correlated even after conditioning on the systematic process (**self exciting nature of defaults**), (AGS10).
Difficult problem since defaults cluster...

Figure: Defaults since 1970 (Moody’s)
Part II

Dynamic Default Timing Model
**Basic idea**

Focus on **default times**. Simplest example is an exponential random variable;

\[ P\{\tau \geq t\} = e^{-\lambda t}. \]  

(2)

A bit more complicated is a hazard function;

\[ P\{\tau \geq t\} = e^{-\int_0^t \lambda_s ds}, \text{ where } \tau = \inf\left\{ t > 0 : \int_0^t \lambda_s ds \geq \epsilon \right\}. \]  

(3)

We want to consider a **random hazard** function \( \lambda_s \). We have

\[ P\{\tau \in (t, t + dt) | \tau \geq t, \lambda_s; 0 \leq s \leq t\} \approx \lambda_t dt. \]  

(4)

- The goal is to model the intensity \( \lambda_t \) in a way that takes into account the empirical observations.
Description of the model

- The intensity $\lambda_t$ follows a mean-reverting jump-diffusion process driven by several terms.
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- The first term is an independent name-specific source of risk.
- The second term is a systematic risk that influences all portfolio constituents and generates diffusive correlation between constituent intensities.
- The third term is the default rate in the pool.
Description of the model

- The intensity $\lambda_t$ follows a mean-reverting jump-diffusion process driven by several terms.
- The first term is an independent name-specific source of risk.
- The second term is a systematic risk that influences all portfolio constituents and generates diffusive correlation between constituent intensities.
- The third term is the default rate in the pool.
- The intricate event dependence structure presents mathematical challenges in the analysis of the system.
The model.

Fix an $N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ and consider the following system:

\[
d\lambda^n_t = -\alpha_n(\lambda^n_t - \bar{\lambda}_n)dt + \sigma_n\sqrt{\lambda^n_t}dW^n_t + \beta^n_C dL^N_t + \varepsilon\beta^n_S \lambda^n_t dX_t \tag{5}
\]

\[
dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t \quad t > 0 \tag{6}
\]

\[
L^N_t = \frac{1}{N} \sum_{n=1}^{N} \chi_{\tau^n \leq t}. \tag{7}
\]

where default times are $\tau^n \overset{\text{def}}{=} \inf \left\{ t \geq 0 : \int_{s=0}^{t} \lambda^n_s ds \geq \varepsilon_n \right\}$.

Typical choices for the process $X_t$:

- Ornstein-Uhlenbeck type: $b_0(x) = -\gamma x$ and $\sigma_0(x) = 1$.
- CIR type: $b_0(x) = \kappa(\theta - x)$ and $\sigma_0(x) = \eta \sqrt{x}$.

This is an interacting particle system formulation (mean field theory).
The model: description 2/2

1. Dynamic and bottoms-up formulation.

2. Our goal is to consider the limit as $N \uparrow \infty$.

   **Heterogenous pool** ⇒ count types. The type space is given by

   $$p^n \overset{\text{def}}{=} (\alpha_n, \bar{\lambda}_n, \sigma_n, \beta_n^C, \beta_n^S) \in \mathcal{P} \overset{\text{def}}{=} \mathbb{R}_+^4 \times \mathbb{R}$$

3. We assume that the following limits exist

   $$\pi^N \overset{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{p^n} \rightarrow \pi, \quad \Lambda^\circ_N \overset{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda_n^\circ} \rightarrow \Lambda_\circ$$
Well-posedness of the model.

Proposition

- The previous system has a unique solution such that $\lambda_t^{N,n} \geq 0$ for every $N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ and $t \geq 0$.

- For each $p \geq 1$ and $T \geq 0$,

\[
K_{p,T} \overset{\text{def}}{=} \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[|\lambda_t^n|^p] < \infty
\]  

(9)
Some related literature

- Cvitaniic, Ma, Zhang (2011)
- Ichiba, Fouque (2011)
Part III

Law of Large Numbers
Definitions & Crucial Observation

- We want to understand the limit of $L_t^N$ as $N \to \infty$.

Notice that for all $t \geq 0$

$$L_t^N = 1 - \mu_t^N(\hat{\mathcal{P}}). \tag{10}$$

where for $\hat{p} = (\alpha, \bar{\lambda}, \sigma, \beta^C, \beta^S, \lambda) \in \hat{\mathcal{P}}$,

$$\mu_t^N \overset{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\hat{p}_t^n} \chi_{\{\tau^n > t\}}. \tag{11}$$

We find $\lim_{N \to \infty} L_t^N$ by studying the limit of $\mu_t^N(\cdot)$ in the Skorokhod topology on $P(\hat{\mathcal{P}}^+)$. 
Characterization of the limit.

**Theorem**

\[ \mu^N \to \bar{\mu}. \text{ (more precisely } \lim_{N \to \infty} \mathbb{P} \left\{ d_P(\hat{\mu}, (\mu^N, \bar{\mu}) \geq \delta \right\} = 0), \text{ where} \]

\[
d \langle f, \bar{\mu}_t \rangle = \left\{ \langle L_1 f, \bar{\mu}_t \rangle + \langle Q, \bar{\mu}_t \rangle \langle L_2 f, \bar{\mu}_t \rangle + \langle L_{3}^X f, \bar{\mu}_t \rangle \right\} dt + \varepsilon \left\langle L_4^X f, \bar{\mu}_t \right\rangle dV_t \quad \text{a.s.} \quad (12)
\]

where for \( \hat{\mu} = (\alpha, \bar{\lambda}, \sigma, \beta^C, \beta^S, \lambda) \) we have \( \langle f, \bar{\mu} \rangle = \int_{\hat{\mu}} f(\hat{\mu}) d\mu(d\hat{\mu}) \) and

\[
(\mathcal{L}_1 f)(\hat{\mu}) = \frac{1}{2} \sigma^2 \lambda \frac{\partial^2 f}{\partial \lambda^2}(\hat{\mu}) - \alpha(\lambda - \bar{\lambda}) \frac{\partial f}{\partial \lambda}(\hat{\mu}) - \lambda f(\hat{\mu}) \quad \text{(diffusive part+ killing)}
\]

\[
(\mathcal{L}_2 f)(\hat{\mu}) = \frac{\partial f}{\partial \lambda}(\hat{\mu}) \quad \text{(contagion)}
\]

\[
(\mathcal{L}_3^X f)(\hat{\mu}) = \beta^S \lambda b_0(x) \frac{\partial f}{\partial \lambda}(\hat{\mu}) + \frac{1}{2} \left( \beta^S \lambda \sigma_0(x) \right)^2 \frac{\partial^2 f}{\partial \lambda^2}(\hat{\mu}) \quad \text{(systematic risk)}
\]

\[
(\mathcal{L}_4^X f)(\hat{\mu}) = \beta^S \lambda \sigma_0(x) \frac{\partial f}{\partial \lambda}(\hat{\mu}) \quad \text{(systematic risk)}
\]

\[
Q(\hat{\mu}) = \lambda \beta^C.
\]
Sketch of the proof 1/7.

Procedure:

1. Step 1: Identification of the limiting martingale problem for $\mu^N$.
2. Step 2: The sequence $\{\mu^N\}_{N \in \mathbb{N}}$ is relatively compact in $D_E[0, \infty)$.
4. Step 4: Realize that $\delta_{\bar{\mu}}$ satisfies the limiting martingale problem.

The analysis is complicated due to the square root singularity of the $\lambda-$ process.
Sketch of the proof 2/7: Martingale Problem

\[ \{ X_N : N < \infty \} \iff \{ A_N : N < \infty \} \]

\[ \Downarrow \quad \Downarrow \]

\[ X \iff A \]

**Fact 1.** Under conditions, for “nice functions“ \( \Phi \) we have that

\[ \Phi(X_t^N) - \Phi(X_0^N) - \int_0^t A_N \Phi(X_s^N) ds \xrightarrow{\text{zero mean martingale}} \]

\[ \Phi(X_t^N) - \Phi(X_0^N) - \int_0^t A_N \Phi(X_s^N) ds \quad (14) \]

**Fact 2.**

\[ \lim_{N \to \infty} E \left[ \Phi(X_t^N) - \Phi(X_0^N) - \int_0^t A \Phi(X_s^N) ds \right] = 0 \iff X_N \Rightarrow X. \]

\[ \lim_{N \to \infty} E \left[ \Phi(X_t^N) - \Phi(X_0^N) - \int_0^t A \Phi(X_s^N) ds \right] = 0 \iff X_N \Rightarrow X. \quad (15) \]
Sketch of the proof 3/7.

**Step 1: Characterization.** This amounts to proving that, for sufficiently smooth functions $\Phi(x, \mu) \in B(\mathbb{R} \times \mathcal{P}(\mathcal{P}))$

\[
\lim_{N \to \infty} \mathbb{E} \left[ \Phi(X_t, \mu^N_t) - \Phi(X_0, \mu^N_0) - \int_0^t (A\Phi)(X_r, \mu^N_r)dr - \int_0^t (B\Phi)(X_r, \mu^N_r)d\nu_r \right] = 0
\]  

(16)

where $(A\Phi, B\Phi)$ characterize the operator of the limiting process, namely

\[
(A\Phi)(x, \mu) = \langle L_1 f, \mu \rangle + \langle Q, \mu \rangle \langle L_2 f, \mu \rangle + \langle L_3^x f, \mu \rangle
\]  

(17)

\[
(B\Phi)(x, \mu) = \epsilon \langle L_4^x f, \mu \rangle
\]  

(18)

- Apply Itô formula for jump processes to $\mu^N$ and make the proper identifications.
Law of Large Numbers

**Sketch of the proof 4/7.**

**Step 1.** The previous statement follows from

**Lemma**

For any $\Phi \in S$ and any $t > 0$ we have that

$$
\Phi(X_t, \mu^N_t) = \Phi(X_0, \mu^N_0) + \int_0^t (A\Phi)(X_r, \mu^N_r)dr + \int_0^t (B\Phi)(X_r, \mu^N_r)dV_r + \\
+ \int_0^t (\hat{A}\Phi)(X_r, \mu^N_r)dr + M^n_t.
$$

Moreover, for any $T > 0$, the following limits hold

$$
\lim_{N \to \infty} \mathbb{E} \left[ \int_0^t \left| (\hat{A}\Phi)(X_r, \mu^N_r) \right| dr \right] = 0 \quad \text{and} \quad \lim_{N \to \infty} \sup_{0 \leq t \leq T} \mathbb{E} [M^n_t]^2 = 0.
$$

(19)
Sketch of the proof 5/7.

**Step 1.** Suppose that (only) the \( n \)-th firm defaults at time \( t \). Recall

\[
\langle f, \mu_t^N \rangle = \int_{\hat{\mathbf{p}}} f(\hat{\mathbf{p}}) \mu_t^N(d\hat{\mathbf{p}}) = \frac{1}{N} \sum_{n=1}^{N} f(\hat{\mathbf{p}}_n^t) \chi_{\{\tau_n^t > t\}}. \tag{21}
\]
Sketch of the proof 5/7.

**Step 1.** Suppose that (only) the n-th firm defaults at time t. Recall

\[
\left\langle f, \mu_t^N \right\rangle = \int_{\hat{\mathcal{P}}} f(\hat{p}) \mu_t^N(d\hat{p}) = \frac{1}{N} \sum_{n=1}^{N} f(\hat{p}_t^n) \chi_{\{\tau^n > t\}}. \tag{21}
\]

Then

\[
\left\langle f, \mu_t^N \right\rangle - \left\langle f, \mu_{t-}^N \right\rangle = \mathcal{J}_n^f(t) \tag{22}
\]

where

\[
\mathcal{J}_n^f(t) = \frac{1}{N} \left[ \sum_{n' = 1}^{N} \left\{ f\left( p_{n'}, \lambda_{n'}^t + \frac{\beta C}{N} \right) - f\left( p_{n'}, \lambda_{n'}^t \right) \right\} \chi_{\{\tau^n > t\}} - f(p_n^t, \lambda_{n}^t) \right] \tag{23}
\]

for all \( t \geq 0, N \in \mathbb{N} \) and \( n \in \{1, 2, \ldots, N\} \).
Sketch of the proof 5/7.

**Step 1.** Suppose that (only) the n-th firm defaults at time t. Recall

\[ \left\langle f, \mu_t^N \right\rangle = \int_{\hat{p}} f(\hat{p}) \mu_t^N (d\hat{p}) = \frac{1}{N} \sum_{n=1}^{N} f(\hat{p}_t^n) \chi_{\{\tau^n > t\}}. \]  

Then

\[ \left\langle f, \mu_t^N \right\rangle - \left\langle f, \mu_{t-}^N \right\rangle = J^f_n(t) \]  

where

\[ J^f_n(t) = \frac{1}{N} \left[ \sum_{n'=1}^{N} \left\{ f \left( \hat{p}_t^{n'}, \lambda_t^{n'} + \frac{\beta C}{N} \right) - f \left( \hat{p}_t^{n'}, \lambda_t^{n'} \right) \right\} \chi_{\{\tau^n > t\}} - f \left( \hat{p}_t^n, \lambda_t^n \right) \right] \]

for all \( t \geq 0, \, N \in \mathbb{N} \) and \( n \in \{1, 2, \ldots, N\} \). Define,

\[ \tilde{J}^f_n(t) = \frac{1}{N} \left[ \sum_{n'=1}^{N} \frac{\beta C}{N} \frac{\partial f}{\partial \lambda} (\hat{p}_t^{n'}) \chi_{\{\tau^n > t\}} - f \left( \hat{p}_t^n, \lambda_t^n \right) \right] \]

Notice that \( |J^f_n(t) - \tilde{J}^f_n(t)| \leq \frac{C}{N^2} \).
Sketch of the proof 6/7.

Step 1. Then

\[
\int_0^t (\hat{A}^n \Phi)(X_r, \mu_r^N)dr \approx O \left( \int_0^t \sum_{n=1}^N \lambda_n^r \mathcal{J}_n^f(r) \chi_{\{\tau^n > r\}} dr - \int_0^t \sum_{n=1}^N \lambda_n^r \tilde{\mathcal{J}}_n^f(r) \chi_{\{\tau^n > r\}} dr \right) \approx 0
\]

(25)

Since

\[
\sum_{n=1}^N \lambda_n^r \tilde{\mathcal{J}}_n^f(r) \chi_{\{\tau^n > r\}} = \sum_{n=1}^N \lambda_n^r \frac{1}{N} \left[ \beta_n^C \sum_{m=1}^N \frac{1}{N} \frac{\partial f}{\partial \lambda} (\hat{p}_r^m) \chi_{\{\tau^n > t\}} - f(\hat{p}_r^n) \right] \chi_{\{\tau^n > r\}}
\]

\[
= \sum_{n=1}^N \lambda_n^r \frac{1}{N} \left[ \beta_n^C \left\langle \mathcal{L}_2 f, \mu_r^N \right\rangle - f(\hat{p}_r^n) \right] \chi_{\{\tau^n > r\}}
\]

\[
= \frac{1}{N} \sum_{n=1}^N \lambda_n^r \beta_n^C \chi_{\{\tau^n > r\}} \left\langle \mathcal{L}_2 f, \mu_r^N \right\rangle - \frac{1}{N} \sum_{n=1}^N \lambda_n^r f(\hat{p}_r^n) \chi_{\{\tau^n > r\}}
\]

\[
= \left\langle \mathcal{Q}, \mu_r^N \right\rangle \left\langle \mathcal{L}_2 f, \mu_r^N \right\rangle - \left\langle \mathcal{L} f, \mu_r^N \right\rangle
\]

(26)

we get the nonlinear term.
Sketch of the proof 7/7

- **Step 2: Existence.** The sequence $\{\mu^N\}_{N \in \mathbb{N}}$ is relatively compact in $D_E[0, \infty)$.

- **Step 3: Uniqueness.** Duality arguments.

- **Step 4: Identification.** Identify the unique solution to the limiting martingale problem; nonlinearity $\implies$ fixed point arguments.
Limiting density.

Thus, the density of the loss distribution, $\bar{\mu}_t = \nu(t, \lambda)d\lambda$, satisfies the nonlinear SPDE (homogeneous case for simplicity):

$$
d\nu(t, \lambda) = \left\{ \mathcal{L}^*_1 \nu(t, \lambda) + \mathcal{L}^*_3, X_t \nu(t, \lambda) + \beta^C \left( \int_0^\infty \lambda \nu(t, \lambda)d\lambda \right) \mathcal{L}^*_2 \nu(t, \lambda) \right\} dt + \mathcal{L}^*_4, X_t \nu(t, \lambda)dV_t, \lambda > 0
$$

$$
\nu(0, \lambda) = \Lambda_{\circ}(\lambda),
\nu(t, 0) = \lim_{\lambda \searrow 0} \nu(t, \lambda) = 0
$$

(27)

where $\mathcal{L}^*_i$ are adjoint operators to $\mathcal{L}_i$, for $i = 1, 2, 3, 4$ respectively.
Limiting density.

Thus, the density of the loss distribution, $\mu_t = \nu(t, \lambda) d\lambda$, satisfies the nonlinear SPDE (homogeneous case for simplicity):

\[
d\nu(t, \lambda) = \left\{ \mathcal{L}_1^* \nu(t, \lambda) + \mathcal{L}_3^* X_t \nu(t, \lambda) + \beta^C \left( \int_0^\infty \lambda \nu(t, \lambda) d\lambda \right) \mathcal{L}_2^* \nu(t, \lambda) \right\} dt + \mathcal{L}_4^* X_t \nu(t, \lambda) dV_t, \lambda > 0
\]

\[
\nu(0, \lambda) = \Lambda_\circ (\lambda), \quad \nu(t, 0) = \lim_{\lambda \to \infty} \nu(t, \lambda) = 0
\]

where $\mathcal{L}_i^*$ are adjoint operators to $\mathcal{L}_i$, for $i = 1, 2, 3, 4$ respectively.

- Not standard SPDE: nonlinear, in the half line, degeneracy at the boundary $\lambda = 0$. 
Part IV

Numerical Solution—Law of Large Numbers
Numerical solution.

Note that the limiting loss \( L_t = 1 - u_0(t) \). Consider the moments

\[
  u_k(t) = \int_0^\infty \lambda^k v(t, \lambda) d\lambda
\]  

Then, some algebra shows that \( u_k(t) \) satisfy the system of SDE’s:

\[
du_k(t) = \left\{ u_k(t) \left( -\alpha + \beta S b_0(X_t)k + 0.5(\beta^2)\sigma^2_0(X_t)k(k-1) \right) \\
+ u_{k-1}(t) \left( 0.5\sigma^2 k(k-1) + \alpha \lambda k + \beta^C ku_1(t) \right) - u_{k+1}(t) \right\} dt + \beta^S \sigma_0(X_t)ku_k(t) dV_t,
\]

\[
u_k(t = 0) = \int_0^\infty \lambda^k \Lambda_o(\lambda) d\lambda.
\]  

Truncate the system of SDE’s at some level and solve it backwards!
Accuracy of numerical solution.

**Figure:** Comparison of distributions of limiting portfolio loss $L_t$ (using 16 moments) and portfolio loss of finite system $L^N_t$ for different $N$ at $t = 1$. 25,000 Monte Carlo trials were used for the finite system and 100,000 for the asymptotic solution. The parameter case is $\sigma = .9$, $\alpha = 4$, $\bar{\lambda} = .2$, $\lambda_0 = .2$, $\beta^C = 4$, and $\beta^S = 8$.

Convergence tends to be faster for larger $\beta^S$. 
Numerical solution.

Figure: Evolution of distribution of limiting portfolio loss $L_t$ over time $t$ (same parameters).

1. Simultaneously for all time horizons.
2. Distribution is not monotonic: widens and then tightens.
Value-At-Risk (VaR) Calculations.

\[ \text{VaR}_\alpha(L) = \inf \{ \ell \in \mathbb{R} : P(L > \ell) \leq 1 - \alpha \} \]  

(30)

Figure: Comparison of \( \text{VaR}_{0.95}(L_t) \) and \( \text{VaR}_{0.99}(L_t) \) (using 16 moments) with \( \text{VaR}_{0.95}(L_t^N) \) and \( \text{VaR}_{0.99}(L_t^N) \) for different \( N \) and several horizons \( t \) (same parameters).

- Simultaneously for all time horizons.
Effect of contagion.

Figure: Comparison of distribution of limiting portfolio loss $L_t$ for different values of the contagion sensitivity $\beta^C$ at $t = 1$ (same parameters).

As contagion increases:

1. Mean of the loss distribution increases.
2. Variance increases.
3. Possibility of large losses increases.
Effect of systematic risk.

Figure: Comparison of distribution of limiting portfolio loss $L_t$ for different values of the systematic risk sensitivity $\beta^S$ at $t = 1$ (same parameters).
Part V

Gaussian Correction
Gaussian correction—a better approximation.

We are interested in the limit as $N \to \infty$ of

$$
\xi^N_t = \sqrt{N} (L^N_t - L_t)
$$

(31)

As in the case of the LLN it is more convenient mathematically to look at empirical measures

$$
\Xi^N_t \overset{\text{def}}{=} \sqrt{N} (\mu^N_t - \bar{\mu}_t)
$$

(32)

Then, it is easy to see that

$$
\xi^N_t = -\Xi^N_t (\hat{P}), \quad t \geq 0,
$$

which implies the second-order approximation to the limiting loss of the portfolio for large $N$

$$
L^N_t \approx L_t - \frac{1}{\sqrt{N}} \Xi_t (\hat{P}).
$$

(33)

where $\Xi$ is the limit of $\Xi^N_t$ in the appropriate sense.
Topological difficulties.

- The situation here is more complicated though compared to the LLN.
- Even though the process \( \{ \Xi^N \} \) is a signed-measure valued process, its limit process \( \{ \Xi \} \) is distribution valued in the appropriate space.
- One needs to consider the equation in weighted Sobolev spaces and several technical difficulties arise, mainly due to the growth and degeneracies of the coefficients which make it difficult to identify the correct weights to use.
Definition of operators.

Let

\[ (G_{x,\mu} f)(\hat{p}) = (L_1 f)(\hat{p}) + (L_3 f)(\hat{p}) + \langle Q_1, \mu \rangle_E (L_2 f)(\hat{p}) + \langle L_2 f, \mu \rangle_E Q_1(\hat{p}) \] (34)

and

\[ (L_5(f, g))(\hat{p}) = \sigma^2 \frac{\partial f}{\partial \lambda}(\hat{p}) \frac{\partial g}{\partial \lambda}(\hat{p}) \lambda \]
\[ (L_6(f, g))(\hat{p}) = f(\hat{p})g(\hat{p}) \lambda \] (35)
\[ (L_7 f)(\hat{p}) = \beta^C f(\hat{p}) \lambda \]
\[ Q_2(\hat{p}) = \lambda (\beta^C)^2. \]
The family \( \{ \Xi_t^N, t \in [0, T] \}_{N \in \mathbb{N}} \) is relatively compact in the space \( D_{\mathcal{W}^*[0, T]} \). Hence, for any subsequence of \( \{ \Xi_t^N, t \in [0, T] \}_{N \in \mathbb{N}} \), there exists a subsubsequence that converges in distribution with limit \( \{ \bar{\Xi}_t, t \in [0, T] \} \). With probability 1, for any \( f \in \mathcal{W} \) any accumulation point \( \bar{\Xi} \) satisfies the unique stochastic evolution equation

\[
\langle f, \bar{\Xi}_t \rangle = \langle f, \bar{\Xi}_0 \rangle + \int_0^t \langle G_{X_s} \bar{\mu}_s f, \bar{\Xi}_s \rangle \, ds + \int_0^t \langle \mathcal{L}_4 X_s \bar{\Xi}_s, \bar{\Xi}_s \rangle \, dV_s + \langle f, \bar{M}_t \rangle, \text{ a.s.}
\]

(36)

where \( \bar{M}_t \) is a distribution-valued martingale, which conditional on the \( \sigma \)-algebra \( \mathcal{V}_t \), \( \bar{M}_t \) is a centered Gaussian with covariance function, for \( f, g \in \mathcal{W} \),

\[
\text{Cov} \left[ \langle f, \bar{M}_t \rangle, \langle g, \bar{M}_t \rangle \bigg| \mathcal{V}_t \right] = \mathbb{E} \left[ \int_0^t \left[ \langle \mathcal{L}_5(f, g), \bar{\mu}_s \rangle + \langle \mathcal{L}_6(f, g), \bar{\mu}_s \rangle + \langle \mathcal{L}_2 f, \bar{\mu}_s \rangle \langle \mathcal{L}_2 g, \bar{\mu}_s \rangle \langle Q_2, \bar{\mu}_s \rangle 
\right.

- \langle \mathcal{L}_7 g, \bar{\mu}_s \rangle \langle \mathcal{L}_2 f, \bar{\mu}_s \rangle - \langle \mathcal{L}_7 f, \bar{\mu}_s \rangle \langle \mathcal{L}_2 g, \bar{\mu}_s \rangle - \langle \mathcal{L}_7 f, \bar{\mu}_s \rangle \langle \mathcal{L}_7 g, \bar{\mu}_s \rangle \rangle \bigg| \mathcal{V}_t \right].
\]

(37)
Sketch of the proof

Procedure:

1. Step 1: Identification of the limiting martingale problem for $\mu^N$.
2. Step 2: Identification of the minimal Sobolev space $W$ such that
   - Step 3: The sequence $\{\Xi^N\}_{N \in \mathbb{N}}$ is relatively compact in $D_{W^*}[0, T]$.
   - Step 4: Uniqueness of the solution to the limiting equation in the appropriate Sobolev space.

- The analysis is complicated due to the growth of the coefficients and the square root singularity of the $\lambda-$ process. Identification of the correct space is essential.
Part VI

Numerical Solution-Central Limit Theorem
**Numerical solution.**

Reduce the stochastic evolution equation to a system of SDE's using the method of moments. Let

$$v_k(t) = \int_0^\infty \lambda^k \Xi_t(d\lambda)$$  \hspace{1cm} (38)

$v_k(t)$ satisfy a system that is not closed, so we truncate at some level $K$.

$$d\mathbf{v}(t) = A(t)\mathbf{v}(t)dt + B\mathbf{v}dX_t + d\mathbf{\tilde{M}}(t),$$

$$\mathbf{v}(t = 0) = \mathbf{v}_0,$$  \hspace{1cm} (39)

where $\mathbf{\tilde{M}}_k(t) = \langle \lambda^k, \mathbf{\tilde{M}}(t) \rangle$ and $[\mathbf{\tilde{M}}_k(t), \mathbf{\tilde{M}}_j(t)] = (\Sigma_M(t))_{kj}$

The fundamental solution $\Psi : [0, T] \times \Omega \rightarrow \mathbb{R}^{K+1,K+1}$ satisfies

$$d\Psi(t) = A(t)\Psi(t)dt + B\Psi(t)dX_t,$$

$$\Psi(t = 0) = I,$$  \hspace{1cm} (40)
Figure: Comparison of approximate loss distribution and actual loss distribution in the finite system at $T = .5$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 1$, and $\beta^S = 1$. $X_t$ is an OU process with mean 1, reversion speed 2, and volatility 1.
Empirical CDF for $\beta^S = \beta^C = 1$ at $T = .5$

Approximation, $N = 100$
Finite System, $N = 100$
<table>
<thead>
<tr>
<th>Time</th>
<th>0.35</th>
<th>0.4</th>
<th>0.45</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acutal 99% VaR</td>
<td>0.09</td>
<td>0.1</td>
<td>0.11</td>
<td>0.12</td>
</tr>
<tr>
<td>Actual 95% VaR</td>
<td>0.13</td>
<td>0.14</td>
<td>0.15</td>
<td>0.16</td>
</tr>
<tr>
<td>Second−Order Approximation of 95% VaR</td>
<td>0.17</td>
<td>0.18</td>
<td>0.19</td>
<td>0.2</td>
</tr>
<tr>
<td>Second−Order Approximation of 99% VaR</td>
<td>0.21</td>
<td>0.22</td>
<td>0.23</td>
<td>0.24</td>
</tr>
<tr>
<td>LLN 95% VaR</td>
<td>0.25</td>
<td>0.26</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>LLN 99% VaR</td>
<td>0.29</td>
<td>0.3</td>
<td>0.31</td>
<td>0.32</td>
</tr>
</tbody>
</table>

**Figure:** Value At Risk for $T=0.5$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 1,$ and $\beta^S = 1$. $X_t$ is an OU process with mean 1, reversion speed 2, and volatility 1.
**Figure:** Call option on the portfolio loss $f(L^N_T) = (L^N_T - S)^+$. Comparison of standard error for direct simulation versus second order approximation. Parameter case is strike $S = 0.12$, $K = 6$, truncation level, $T = 0.5$, $\sigma = .9$, $\alpha = 4$, $\lambda_0 = .2$, $\bar{\lambda} = .2$, $\beta^C = 1$, and $\beta^S = 1$. $X_t$ is an OU process with mean 1, reversion speed 2, and volatility 1.
Part VII

Summary
Summary & Discussion

1. We have developed a dynamic default timing model which is empirically motivated and amenable to analysis.

2. We characterized the behavior of the default rate in the pool as its size goes to infinity.

3. We studied law of large numbers and second order Gaussian corrections in the pool and showed that the model captures several different behaviors.

4. LLN and CLT are characterized by stochastic evolution equations that can be solved numerically via a method of moments.

5. Calculate important risk measures and analyze the behavior of the portfolio → better risk management.
Part VIII

References
References


References


References


Konstantinos Spiliopoulos, Justin A. Sirignano and Kay Giesecke, A Conditionally Gaussian Approximation for Large Credit Portfolios, in preparation.


Thank You!!!!