Optimal Stopping Problems for Spectrally Negative Lévy Processes and Applications in Finance

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Outline

- **Preliminaries:**
  Scale functions, Resolvent (Potential) Measures, and Optimal Stopping.

- **Part 1:**
  First Order Condition and Continuous/Smooth Fit Condition (joint with M. Egami).

- **Part 2:**
  Applications to Multiple Stopping.

- **Part 3:**
  Applications to Leland’s Capital Structure Model (joint with B.A. Surya).
Preliminaries
Spectrally Negative Lévy Process

Defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), let \(X = \{X_t; t \geq 0\}\) be a spectrally negative Lévy process, i.e.

1. The paths are almost surely right continuous with left limits.
2. For \(0 \leq s \leq t\), \(X_t - X_s\) is equal in distribution to \(X_{t-s}\).
3. For \(0 \leq s \leq t\), \(X_t - X_s\) is independent of \(\{X_u : u \leq s\}\).
4. Jumps are almost surely negative (spectrally negative).

Laplace exponent is the logarithm of its Laplace transform:

\[
\psi(s) := \log \mathbb{E}^0 \left[ e^{sX_1} \right], \quad s \in \mathbb{C}.
\]

and we can write

\[
\psi(s) := cs + \frac{1}{2} \sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx1_{\{0<x<1\}}) \Pi(dx), \quad s \in \mathbb{C}.
\]
Optimal Stopping Problems

Let $F$ be the filtration generated by $X$ and $S$ be a set of $F$-stopping times.

We shall consider a general optimal stopping problem of the form:

$$u(x) := \sup_{\tau \in S} \mathbb{E}^x \left[ e^{-q\tau} g(X_{\tau}) + \int_0^\tau e^{-qt} h(X_t) \, dt \right],$$

for every $x \in \mathbb{R}$ and some discount factor $q > 0$ and locally-bounded measurable functions $g, h : \mathbb{R} \mapsto \mathbb{R}$ which represent, respectively, the payoff received at a given stopping time $\tau$ and the running reward up to $\tau$. 
Opt. Stopping for SN Lévy

Optimal Stopping:
- Avram & Kyprianou & Pistorius (AAP, 2004); Alili & Kyprianou (AAP, 2005); Kyprianou & Surya (ECP, 2005); Kyprianou & Surya (F&S, 2007); Egami and Y. (Stochastics, forthcoming); Leung & Y. (QF, forthcoming).

Stopping Games:
- Baurdoux & Kyprianou, (EJP, 2008); Baurdoux & Kyprianou, (TPA, 2009); Baurdoux & Kyprianou & Pardo (SPA, 2011); Egami & Leung & Y. (SPA, forthcoming).

Optimal Control (optimal dividend):
- Loeffen (AAP, 2008); Kyprianou & Palmowski (JAP, 2007).
Scale Functions

Associated with every spectrally negative Lévy process, there exists a 
(q-)scale function

\[ W^{(q)} : \mathbb{R} \rightarrow \mathbb{R}, \]

whose Laplace transform is given by

\[ \int_{0}^{\infty} e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}, \quad \beta > \Phi(q) \]

where \( \Phi(q) \) is the (largest) positive root of

\[ \Phi(q) := \sup \{ s \geq 0 : \psi(s) = q \}. \]

We assume \( W^{(q)}(x) = 0 \) on \((-\infty, 0)\).
Scale Functions (Cont’d)

We define the first down- and up-crossing times, respectively, by

\[
\tau_a := \inf \{ t \geq 0 : X_t < a \},
\]
\[
\tau_b^+ := \inf \{ t \geq 0 : X_t > b \}
\]

for any \( 0 \leq a < x < b \). Then we have, for every \( 0 \leq x < b \),

\[
\mathbb{E}^x \left[ e^{-q\tau_b^+} 1 \{ \tau_b^+ < \tau_0 \} \right] = \frac{W(q)(x)}{W(q)(b)},
\]
\[
\mathbb{E}^x \left[ e^{-q\tau_0} 1 \{ \tau_b^+ > \tau_0 \} \right] = Z(q)(x) - Z(q)(b) \frac{W(q)(x)}{W(q)(b)}
\]

where \( Z(q)(x) = 1 + q \int_0^x W(q)(y)dy \) for every \( x \in \mathbb{R} \).
Potential & Compensation

- For any measurable function $h$, we have

$$
\mathbb{E}^x \left[ \int_0^{\tau_0} e^{-qt} h(X_t) dt \right]
= W(q)(x) \int_0^\infty e^{-\zeta_q y} h(y) dy - \int_0^x W(q)(x - y) h(y) dy.
$$

- Given a random time-space function $\phi = \phi(t, x)[\omega]$, $\mathbb{E} \left[ \int_{[0,t]} \int_\mathbb{R} \phi(s, x) N(ds \times dx) \right] = \mathbb{E} \left[ \int_{[0,t]} \int_\mathbb{R} \phi(s, x) ds \Pi(dx) \right]

if
- for each $t \geq 0$, $\phi(t, x)[\omega]$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable,
- for each $x \in \mathbb{R}$, $\phi(t, x)[\omega]$ is a.s. left continuous.
Overshoots and Undershoots

For all $B \in \mathcal{B}(0, \infty)$ and $A \in \mathcal{B}(-\infty, 0)$,

$$E^x \left[ e^{-q \tau_0} 1\{X_{\tau_0-} \in B, X_{\tau_0} \in A, \tau_0 < \infty\} \right]$$

$$= \int_0^\infty \Pi(du) \left\{ W^{(q)}(x) \int_{B \cap (A+u)} e^{-\zeta q y} dy - \int_{B \cap (A+u)} dy W^{(q)}(x - y) \right\}.$$
Part 1: Continuous/Smooth Fit vs. First-Order Condition
Let $F$ be the filtration generated by $X$ and $S$ be a set of $F$-stopping times.

We shall consider a general optimal stopping problem of the form:

$$u(x) := \sup_{\tau \in S} \mathbb{E}^x \left[ e^{-q\tau} g(X_\tau) + \int_0^\tau e^{-qt} h(X_t) dt \right],$$

for every $x \in \mathbb{R}$ and some discount factor $q > 0$ and locally-bounded measurable functions $g, h : \mathbb{R} \mapsto \mathbb{R}$ which represent, respectively, the payoff received at a given stopping time $\tau$ and the running reward up to $\tau$. 
Optimal Stopping Problems

Define the first down-crossing time of the form

\[ \tau_A := \inf \{ t > 0 : X_t \leq A \} , \quad A \in \mathbb{R} . \]

Let us denote the corresponding expected payoff by

\[ u_A(x) := \mathbb{E}^x \left[ e^{-q\tau_A} g(X_{\tau_A}) + \int_0^{\tau_A} e^{-qt} h(X_t) dt \right] , \]

where

\[ = \begin{cases} 
\Gamma_1(x; A) + \Gamma_2(x; A) + \Gamma_3(x; A), & x > A, \\
g(x), & x \leq A, 
\end{cases} \]

where, for every \( x > A \),

\[ \Gamma_1(x; A) := g(A) \mathbb{E}^x \left[ e^{-q\tau_A} \right] , \]

\[ \Gamma_2(x; A) := \mathbb{E}^x \left[ e^{-q\tau_A} (g(X_{\tau_A}) - g(A))1 \{ X_{\tau_A} < A, \tau_A < \infty \} \right] , \]

\[ \Gamma_3(x; A) := \mathbb{E}^x \left[ \int_0^{\tau_A} e^{-qt} h(X_t) dt \right] . \]
Optimal Stopping Problems

For every $x > A$, we have

$$
\Gamma_1(x; A) := g(A) \mathbb{E}^x \left[ e^{-q \tau_A} \right]
= g(A) \left[ Z(q)(x - A) - \frac{q}{\Phi(q)} W(q)(x - A) \right],
$$

$$
\Gamma_2(x; A) := \mathbb{E}^x \left[ e^{-q \tau_A} (g(X_{\tau_A}) - g(A)) \mathbb{1}_{\{X_{\tau_A} < A, \tau_A < \infty\}} \right],
= \int_0^\infty \Pi(du) \left[ W(q)(x - A) \int_0^u e^{-\Phi(q)y}(g(y + A - u) - g(A))dy 
- \int_0^{u \wedge (x - A)} W(q)(x - z - A)(g(z + A - u) - g(A))dz \right],
$$

$$
\Gamma_3(x; A) := \mathbb{E}^x \left[ \int_0^{\tau_A} e^{-qt} h(X_t)dt \right]
= W(q)(x - A) \int_0^\infty e^{-\Phi(q)y} h(y + A)dy - \int_A^x W(q)(x - y)h(y)dy.
$$
**First-Order Condition**

Fix $x > A$. Suppose

1. $g$ is $C^2$ in some neighborhood of $A$
2. $g$ satisfies $\int_1^\infty \Pi(du) \max_{A-u \leq \zeta \leq A} |g(\zeta) - g(A)| < \infty$, 
3. $\int_1^\infty \Pi(du) \sup_{0 \leq \xi \leq \delta} |g(A + \xi) - g(A + \xi - u)| < \infty$ for some $\delta > 0$. 

Then, we have 

$$\frac{\partial}{\partial A} u_A(x) = -\Theta^{(q)}(x - A) \Lambda(A),$$

where 

$$\Theta^{(q)}(y) := e^{\Phi(q)y} W_{\Phi(q)}'(y) \text{ with } W_{\Phi(q)}'(y) := e^{-q \Phi(q)} W(q)(x),$$

$$\Lambda(A) := -\frac{q}{\Phi(q)} g(A) - \frac{\sigma^2}{2} g'(A) + \rho^{(q)}_{g, A} + \int_0^\infty e^{-\Phi(q)y} h(y + A)dy,$$

$$\rho^{(q)}_{g, A} := \int_0^\infty \Pi(du) \int_0^u e^{-\Phi(q)z}(g(z + A - u) - g(A))dz,$$
First-Order Condition (Cont’d)

We obtain

\[ \Lambda(A) \leq (\geq)0 \implies \frac{\partial}{\partial A} u_A(x) \geq (\leq)0 \quad \forall x > A. \]

If there exists \( A^* \) such that

\[ \Lambda(A^*) = 0, \]

then the stopping time \( \tau_{A^*} \) naturally becomes a reasonable candidate for the optimal stopping time.
Continuous Fit

Fix $A \in \mathbb{R}$ and suppose

1. $g$ is $C^2$ in some neighborhood of $A$
2. $g$ satisfies $\int_{1}^{\infty} \Pi(du) \max_{u-\zeta \leq A} |g(\zeta) - g(A)| < \infty$.

1. If $X$ is of bounded variation, the continuous fit condition $u_A(A+) = g(A)$ holds if and only if

   $$\Lambda(A) = 0.$$ 

2. If $X$ is of unbounded variation (including the case $\sigma = 0$), it is automatically satisfied.
Fix $A \in \mathbb{R}$ and suppose

1. Suppose $\sigma > 0$.
2. $g$ is $C^2$ in some neighborhood of $A$
3. $g$ satisfies $\int_1^\infty \prod (du) \max_{A-u \leq \zeta \leq A} |g(\zeta) - g(A)| < \infty$.

Then, the smooth fit condition $u'_A(A+) = g'(A)$ holds if and only if $\Lambda(A) = 0$. 

Summary of Conditions

- The first-order condition is $\Lambda(A) = 0$ where

$$
\Lambda(A) := -\frac{q}{\Phi(q)}g(A) - \frac{\sigma^2}{2}g'(A) + \rho_{g,A}^{(q)} + \int_0^\infty e^{-\Phi(q)y}h(y + A)dy,
$$

$$
\rho_{g,A}^{(q)} := \int_0^\infty \Pi(du) \int_u^\infty e^{-\Phi(q)z}(g(z + A - u) - g(A))dz.
$$

- The Continuous/Smooth fit conditions are given below:

<table>
<thead>
<tr>
<th></th>
<th>Continuous-fit</th>
<th>Smooth-fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded var.</td>
<td>$\Lambda(A) = 0$</td>
<td>N/A</td>
</tr>
<tr>
<td>$\sigma &gt; 0$</td>
<td>Automatically satisfied</td>
<td>$\Lambda(A) = 0$</td>
</tr>
</tbody>
</table>

- The first-order condition and the smooth/continuous fit condition are indeed equivalent.
The problem of pricing a perpetual American option is a special case with

\[ g(x) = K - e^x \quad \text{and} \quad h \equiv 0. \]

where \( e^x \) models the stock price and \( K > 0 \) is the strike price.

This classical problem is known to have an optimal stopping time of threshold type (Mordecki 2002).

In this case, \( \Lambda(A) \) is monotone, and the optimal threshold level is given by

\[ A^* = \log \left( K \frac{q}{\Phi(q)} \frac{\Phi(q) - 1}{q - \psi(1)} \right). \]
Part 2:
Application to Multiple Stopping
Multiple Stopping

\[ U^{(M)}(x) := \sup_{(\tau^{(1)}, \ldots, \tau^{(M)}) \in S_M} \sum_{m=1}^{M} \mathbb{E}^x \left[ \int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt} F_m(X_t) dt + e^{-r\tau^{(m)}} g_m(X_{\tau^{(m)}}) 1_{\{\tau^{(m)} < \infty\}} \right], \]

\[ \tilde{U}^{(M)}(x) := \sup_{(\tau^{(1)}, \ldots, \tau^{(M)}) \in \tilde{S}_M} \sum_{m=1}^{M} \mathbb{E}^x \left[ \int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt} F_m(X_t) dt + e^{-r\tau^{(m)}} g_m(X_{\tau^{(m)}}) 1_{\{\tau^{(m)} < \infty\}} \right], \]

where \( \tau^{(0)} := 0 \) and

\[ S_M := \{ \tau^{(m)} \in S, 1 \leq m \leq M : \tau^{(1)} \leq \cdots \leq \tau^{(M)} \}, \]

\[ \tilde{S}_M := \{ \tau^{(m)} = \tau_{A_m} \in \tilde{S}, 1 \leq m \leq M : A_1 \geq A_2 \cdots \geq A_M \}. \]
Main Results 1

Regarding the running reward function $F$, define the differences:

$$f_m := F_m - F_{m+1}, \quad 1 \leq m \leq M,$$

with $F_{M+1} \equiv 0$. Suppose

1. $f(\cdot)$ is continuous, increasing, $f(-\infty) := \lim_{x \downarrow -\infty} f(x) > -\infty$ and $\int_0^\infty e^{-\Phi(r)y} f_+(y)dy < \infty$;

2. $g_m(x) := K_m - \sum_{i=1}^{N_m} c_m e^{a_m x}$, $1 \leq m \leq M$, for some $K \in \mathbb{R}$ and strictly positive constants $a_i$ and $c_i$, $1 \leq i \leq N$.

Then there is a sequence $A_1^* \geq \cdots \geq A_M^*$, such that $(\tau_{A_1^*}, \ldots, \tau_{A_M^*})$ is optimal over $S_M := \{\tau^{(m)} \in S, 1 \leq m \leq M : \tau^{(1)} \leq \cdots \leq \tau^{(M)}\}$.
Main Results 2

Suppose

1. $f(\cdot)$ is increasing such that $f(-\infty) := \lim_{x \downarrow -\infty} f(x) > -\infty$ and 
   \[ \int_0^\infty e^{-\Phi(r)} f_+(y) dy < \infty; \]
2. $g(\cdot)$ is twice-differentiable, concave and monotonically decreasing.

Then there is a sequence $A_1^* \geq \cdots \geq A_M^*$, such that it is optimal over
\[ \tilde{\mathcal{S}}_M := \{ \tau^{(m)} = \tau_{A_m} \in \tilde{\mathcal{S}}, 1 \leq m \leq M : A_1 \geq A_2 \cdots \geq A_M \}. \]
Define, for any subinterval $\mathcal{I} = \{\min \mathcal{I}, \min \mathcal{I} + 1, \ldots, \max \mathcal{I}\} \subset \{1, \ldots, M\}$,

$$g_{\mathcal{I}} := \sum_{i \in \mathcal{I}} g_i \quad \text{and} \quad f_{\mathcal{I}} := F_{\min \mathcal{I}} - F_{\max \mathcal{I} + 1}.$$ 

We have

$$\Lambda_{\mathcal{I}}(A) := \Lambda(A; f_{\mathcal{I}}, g_{\mathcal{I}}) = \Lambda \left( A; \sum_{m \in \mathcal{I}} f_m, \sum_{m \in \mathcal{I}} g_m \right) = \sum_{m \in \mathcal{I}} \Lambda_m(A)$$

is increasing; let $A^*_{\mathcal{I}}$ be the root of $\Lambda_{\mathcal{I}}(A) = 0$. 
Solutions (Cont’d)

Given $1 \leq m \leq M$, let us partition $\{m, m + 1, \ldots, M\}$ to an $L(m)$ number of (non-empty) disjoint sets $I_m := \{I(k; m), 1 \leq k \leq L(m)\}$ such that

$$\{m, m + 1, \ldots, M\} = I(1; m) \cup \cdots \cup I(L(m); m)$$

where, if $L(m) = 1$, $I(1; m) = \{m, \ldots, M\}$ and, if $L(m) \geq 2$,

$$I(1; m) := \{m, \ldots, n_{1,m} - 1\},$$
$$I(l; m) := \{n_{l-1,m}, \ldots, n_{l,m} - 1\}, \quad 2 \leq l \leq L(m) - 1,$$
$$I(L(m); m) := \{n_{L(m)-1,m}, \ldots, M\},$$

for some integers $m < n_{1,m} < \cdots < n_{L(m)-1,m} < M$. We consider the strategy such that, if $k$ and $l$ are in the same set, then the $k$-th and $l$-th stops occur simultaneously a.s.
Algorithm $\mathcal{I}_{m-1}^* = \text{Update}(\mathcal{I}_m^*, m)$

Step 1 Set $i = 1$.

Step 2 Set

$$\hat{\mathcal{I}} := \left\{ \begin{array}{ll} \{m-1\}, & i = 1, \\ \{m-1\} \cup \mathcal{I}^*(1; m) \cup \cdots \cup \mathcal{I}^*(i-1; m), & i \geq 2. \end{array} \right.$$ 

Step 3 Compute $A_{\hat{\mathcal{I}}}^*$ and

1. if $i = L^*(m) + 1$, then stop and return
   $$\mathcal{I}_{m-1}^* = \{\mathcal{I}^*(1; m-1)\} \text{ with } L^*(m-1) = 1 \text{ and } \mathcal{I}^*(1; m-1) = \{m-1, \ldots, M\};$$

2. if $A_{\hat{\mathcal{I}}}^* > A_{\mathcal{I}^*(i;m)}^*$, then stop and return
   $$\mathcal{I}_{m-1}^* = \{\mathcal{I}^*(k; m-1), 1 \leq k \leq L^*(m-1)\} \text{ with } L^*(m-1) = L^*(m) - i + 2 \text{ and }$$
   $$\mathcal{I}^*(1; m-1) = \hat{\mathcal{I}},$$
   $$\mathcal{I}^*(l; m-1) = \mathcal{I}^*(l+i-2; m), \quad 2 \leq l \leq L^*(m-1);$$

3. if $A_{\hat{\mathcal{I}}}^* \leq A_{\mathcal{I}^*(i;m)}^*$, set $i = i + 1$ and go back to Step 2.
Numerical Examples

For $g$:

(a) mixture of exponential functions: $g^{(\text{exp})}$;

(b) linear function: $g^{(\text{lin})}(x) := -\alpha x + \beta$, $x \in \mathbb{R}$, $\alpha > 0$ and $\beta \in \mathbb{R}$.

For $f$:

(i) simple function: $f^{(\text{sim})}(y) := \sum_{-\infty < n < \infty} f^{(n)} 1_{I_n}(y)$ for some constants $\ldots < f(-2) < f(-1) < f(0) < f(1) < f(2) < \ldots$ such that $-\infty < \lim_{n \to -\infty} f^{(n)} \leq \lim_{n \to \infty} f^{(n)} < \infty$ and subdivisions $I_n := (l_n, l_{n+1}]$ of $\mathbb{R}$;

(ii) linear function with a lower bound: $f^{(\text{lin})}(y) := b_1[(y + b_2) \lor b_3]$ for some $b_1 > 0$ and $b_2, b_3 \in \mathbb{R}$;

(iii) exponential function with an upper bound: $f^{(\text{exp})}(y) := e^{(Ly)^\wedge B}$ for some $L > 0$ and $B \in \mathbb{R}$. 
Numerical Examples (Cont’d)

Case 1

Case 2

Case 3

Case 4
Numerical Examples (Cont’d)

Case 1

Case 2

Case 3

Case 4
Part 3:
Application to Leland-Toft Problem
Tradeoff Theory

- Pros and Cons of Debt Financing:
  - Pros: Tax Benefits
  - Cons: Bankruptcy Costs

- A firm determines its optimal capital structure by striking the balance between
  - maximizing tax benefits
  - minimizing bankruptcy costs.

- Challenges:
  - The Tradeoff Theory tends to overestimate the tax benefits and underestimate the bankruptcy costs.
  - In reality, the observed leverage is much lower than what the Tradeoff Theory implies.
Model with Endogenous Defaults

Leland Model (JF, 1994)

1. A structural approach where the default is triggered by the first time a stochastic process downcrosses a level.
2. A firm is financed by a perpetual debt that is terminated only at bankruptcy.
3. The bankruptcy triggering level is endogenously decided by the equity holder so that the equity value is maximized.

Leland-Toft Model (JF, 1996)

An extension with a constant debt profile modifying 2 in Leland Model (1994) – still analytically tractable.
Empirical Assessment

Leland-Toft (1996):

1. Because the asset value process is assumed to be a geometric Brownian motion, the credit spread converges to zero as the maturity goes to zero. Moreover, the bankruptcy cost can be controlled deterministically.

2. The bankruptcy cost is proportional to the asset value.

3. The tax rebate rate is a constant.

Empirically, 1-3 are not what is observed in reality.

- For 1, non-zero credit spreads can be attained by introducing jumps in the process (details in the next slide).
- We aim to resolve 2 and 3 in this paper.
Introducing Jumps

Advantages of the Lévy model:

- non-zero credit spread achieved as maturity goes to zero.
- the bankruptcy costs is no longer deterministic.

Examples:

- A general spectrally negative process:
  - Hilberink and Rogers (F&S, 2002)
  - Kyprianou and Surya (F&S, 2007)

- Double exponential jump diffusion:
  - Chen and Kou (MF, 2009)
Our Objective

We start at the general spectrally negative model (Kyprianou and Surya (2007)) and relaxes the assumptions that

- the bankruptcy costs is proportional to the asset value,
- tax rate is constant,
Original Model: the Firm

- The firm’s asset value is $V_t := e^{X_t}$, $t \geq 0$ where
  - $r > 0$ is the positive risk-free interest rate,
  - $0 < \delta < r$ is the total payout rate to the firm’s investors.
  - $\{e^{-(r-\delta)t}V_t; t \geq 0\}$ is a martingale.

- The firm is partly financed by debt with a constant debt profile;
  - it issues new debt at a constant rate $p$ with maturity profile $\varphi(s) := me^{-ms}$;
  - in the time interval $(t, t + dt)$, it issues debt with face value $p\varphi(s)dt ds$ that matures in time interval $(t + s, t + s + ds)$.

- The face value of all debt is a constant value,

\[
P := \int_0^\infty pe^{-ms} ds = \frac{p}{m}.
\]
Original Model: Debt Value

- The bankruptcy time is modeled by:
  \[
  \tau_B^- := \inf \{ t \geq 0 : X_t < B \}, \quad B \in \mathbb{R}.
  \]

- The value of the debt with a unit face value and maturity \( t > 0 \):
  \[
  d(x; B, t) := \mathbb{E}_x \left[ \int_0^{t \land \tau_B^-} e^{-rs} \hat{\rho} ds \right] + \mathbb{E}_x \left[ e^{-rt} 1_{\{t < \tau_B^-\}} \right] + \frac{1}{P} \mathbb{E}_x \left[ e^{-r\tau_B^- + X_{\tau_B^-}} (1 - \hat{\eta}) 1_{\{\tau_B^- < t\}} \right].
  \]

- The total value of debt becomes
  \[
  \mathcal{D}(x; B) := \int_0^{\infty} p e^{-mt} d(x; B, t) dt
  = \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-(r+m)t} (P\hat{\rho} + p) dt \right] + \mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} (1 - \hat{\eta}) \right].
  \]
Original Model: Firm Value

The firm value is

\[ V(x; B) := e^x + \mathbb{E}_x \left[ \int_0^{\tau_B^{-}} e^{-rt} P\hat{\gamma}\hat{\rho} dt \right] - \hat{\eta}\mathbb{E}_x \left[ e^{-r\tau_B^{-} + X_{\tau_B^{-}}} \right], \]

where \( \hat{\gamma} \) is a corporate tax rate.
The problem is to pursue an optimal bankruptcy level $B \in \mathbb{R}$ that maximizes the equity value,

$$\mathcal{E}(x; B) := V(x; B) - D(x; B), \quad x > B,$$

subject to the limit liability constraint,

$$\mathcal{E}(x; B) \geq 0, \quad x \geq B,$$

if such a level exists.
Nonlinearity of Bankruptcy Costs

Figure: Reproductions of Figures 1 and 2 of Warner (Journal of Finance, 1977). Based on the 11 railroad firms that went bankrupt between 1933 and 1955.
Nonlinearity of Bankruptcy Costs

Quotes from Warner (1977):

1. “While the higher market-value railroads generally did incur higher bankruptcy costs, the costs do not appear to be directly proportional to market value.”

2. “The percentage appears to decline for railroads with relatively high market values”.

3. “This evidence suggests that there are substantial fixed costs associated with the railroad bankruptcy process, and hence economies of scale with respect to bankruptcy costs.”
Nonlinearity of Tax Benefits

As observed by, e.g., Graham and Smith (1998), the effective tax rate is not a constant (i.e. tax as a function of income is not linear).

- Caused mainly by the asymmetry of the tax code: profits are taxed but losses are not taxed – also by carry forwards/backwards.
- Graham and Smith (1998) empirically showed (via simulation) that the effective tax function (total tax as a function of income) is convex.
Some authors consider the tax cutoff level $V_T$:

$$V(x; B) := e^x + \mathbb{E}_x \left[ \int_0^{\tau B} e^{-r t} 1 \{ X_t \geq \log V_T \} \hat{P} \hat{\gamma} \hat{\rho} dt \right] - \hat{\eta} \mathbb{E}_x \left[ e^{\tau B} - e^{\tau B + X_{\tau B}} \right].$$

Here, its (full) rebate on coupon payments is gained if and only if $V_t \geq V_T$ or $X_t \geq \log V_T$ for some given cutoff level $V_T > 0$. 
Tax Cutoff

Quotes from Hilberink and Rogers (2004) about the tax cutoff:

- “Again this is an idealisation, but it is intended to reflect the idea that when the coupons exceed the profits, you are unable to reclaim the tax on the coupon payments in excess of the profits.”

- “Without such a tax cutoff, the numerical values of the coupons become ridiculously high, with the firm in effect promising huge returns financed by tax rebates; the tax cutoff prevents this.”
Our Generalization

Our Model:

- **Bankruptcy Costs:** \( \hat{\eta} \exp(X_{\tau_B^-}) \rightarrow \overline{\eta}(X_{\tau_B^-}) \exp(X_{\tau_B^-}) = \eta(X_{\tau_B^-}) \)
  
  - \( \overline{\eta}(x) := e^{-x}\eta(x) \) is the ratio of bankruptcy costs relative to the asset value.

- **Tax rebates:** \( P\hat{\gamma}\hat{\rho} \rightarrow f(X_t) \).

- **Coupon:** \( \hat{\rho} \rightarrow \rho(X_t) \).
Generalized Debt Value

Original:

\[ D(x; B) = \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-(r+m)t} (P\rho + p) \, dt \right] 
+ \mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} (1 - \eta) \right]. \]

Our generalization:

\[ D(x; B) = \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-(r+m)t} (P\rho(X_t) + p) \, dt \right] 
+ \mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} (1 - \eta(X_{\tau_B^-})) \right]. \]
Generalized Firm Value

- **Original:**

\[
\mathcal{V}(x; B) := e^x + \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-rt} P\hat{\gamma} \hat{\rho} dt \right] - \hat{\eta} \mathbb{E}_x \left[ e^{-r\tau_B^- + X_{\tau_B^-}} \right].
\]

- **Tax benefits**

- **Bankruptcy costs**

- **Our generalization:**

\[
\mathcal{V}(x; B) = e^x + \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-rt} f(X_t) dt \right] - \mathbb{E}_x \left[ e^{-r\tau_B^-} \eta \left( X_{\tau_B^-} \right) \right].
\]
Our Goal

- We assume that $X$ is a spectrally negative Lévy process.
- We obtain $B$ that maximizes

$$
\mathcal{E}(x; B) := \mathcal{V}(x; B) - \mathcal{D}(x; B),
$$

subject to $\mathcal{E}(x; B) \geq 0, \quad x \geq B$. 
Main Results

If there exits $B^*$ such that $K_1^{(r,m)}(B) \geq 0 \iff B \geq B^*$ and $K_2^{(r)}(B) \geq 0$ for every $B \geq B^*$, then $B^*$ is the optimal bankruptcy level.

Here, for all $B \in \mathbb{R}$,

$$K_1^{(r,m)}(B) := \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} e^B - G_1^{(r+m)}(B) + G_2^{(r)}(B) - J^{(r,m)}(B),$$

$$K_2^{(r)}(B) := G_2^{(r)}(B) + \frac{r}{\Phi(r)} \eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2} \eta'(B),$$

$$H^{(q)}(B) := \int_0^\infty \Pi(du) \left\{ \int_0^u e^{-\Phi(q) z} [\eta(B) - \eta(B - (u - z))]dz \right\},$$

$$G_i^{(q)}(B) := \int_0^\infty e^{-\Phi(q) y} f_i(y + B)dy, \quad i = 1, 2,$$

$$J^{(r,m)}(B) := \left( \frac{r + m}{\Phi(r + m)} - \frac{r}{\Phi(r)} \right) \eta(B) - \left( H^{(r)}(B) - H^{(r+m)}(B) \right).$$
Main Results (Cont’d)

The corresponding debt and firm values are given by:

\[
D(x; B) = e^x - e^B \Gamma^{(r+m)}(x - B) + M_1^{(r+m)}(x; B) - \Lambda^{(r+m)}(x; B), \\
V(x; B) = e^x + M_2^{(r)}(x; B) - \Lambda^{(r)}(x; B),
\]

where

\[
\Lambda^{(q)}(x; B) := \eta(B) \left[ Z^{(q)}(x - B) - \frac{q}{\Phi(q)} W^{(q)}(x - B) \right] \\
- W^{(q)}(x - B) H^{(q)}(B) \\
+ \int_0^\infty \Pi(du) \int_0^u W^{(q)}(x - z - B)[\eta(B) - \eta(z + B - u)]dz,
\]

\[
M_i^{(q)}(x; B) := W^{(q)}(x - B) G_i^{(q)}(B) - \int_B^x W^{(q)}(x - y)f_i(y)dy,
\]

\[
\Gamma^{(q)}(y) := \frac{\kappa(1) - q}{1 - \Phi(q)} W^{(q)}(y) + (\kappa(1) - q)e^y \int_0^y e^{-z} W^{(q)}(z)dz.
\]
An Example

Theorem
We suppose (1) $\eta(y)$ is positive and increasing, (2) $\bar{\eta}(y)$ is decreasing, and (3) $f$ is increasing. Then $B^*$ is an optimal bankruptcy level.

In other words, the optimality holds when, monotonically in the asset value,
1. the loss amount at bankruptcy is positive and increasing.
2. its percentage is decreasing,
3. the value of tax benefits is increasing,
Numerical Results

Consider the case

\[
\bar{\eta}(x) = \eta_0 \left( e^{-a(x-b)} \wedge 1 \right), \quad f(x) = P\hat{\gamma}\hat{\rho} \left( e^{x-c} \wedge 1 \right),
\]

and a constant coupon rate.

Regarding \( X \), we consider the case \( \sigma > 0 \) and jumps are of exponential type with Lévy measure

\[
\Pi(du) = \lambda \beta e^{-\beta u}, \quad u > 0.
\]

Its scale function has an analytical expression by Egami and Yamazaki (2010).

We use \( r = 7.5\%, \delta = 7\%, \hat{\gamma} = 35\%, \sigma = 0.2, \lambda = 0.5, \beta = 9 \), and

- case 1: \( \eta_0 = 0.9, a = 0.5, b = 0 \) and \( c = 5 \),
- case 2: \( \eta_0 = 0.5, a = 0.01, b = 5 \) and \( c = 0 \).
Numerical Results (Cont’d)

debt value (case 1)

debt value (case 2)

firm value (case 1)

firm value (case 2)

equity value (case 1)

equity value (case 2)
Two-stage Problem

From the firm’s perspective, they want to choose $P$ such that the firm’s value $\mathcal{V}$ is maximized, namely, for fixed $x$,

$$\max_P \mathcal{V}(x; B^*(P), P).$$

Case 1

Case 2

Figure: We set $V_0 = 100$ (or $x = \log(100)$). The optimal face values of debt are $P^* = 73.7$ and $P^* = 39$. 
References

