Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences

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Joint work with
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Usual approach to derivatives pricing:

1. Model the underlying securities as a $J$-dimensional stochastic process $(R_t)$ on a probability space $(\Omega, \mathcal{F}, P)$
2. Price derivatives by $\mathbb{E}_Q[.]$ for some equivalent martingale measure $Q \sim P$
Motivation

Usual approach to derivatives pricing:

1. Model the underlying securities as a $J$-dimensional stochastic process $(R_t)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2. Price derivatives by $\mathbb{E}_Q[\cdot]$ for some equivalent martingale measure $Q \sim \mathbb{P}$.

In complete markets: $Q$ is unique

binomial tree models, Black–Scholes model ... 

In incomplete markets: $Q$ is not unique

trinomial tree models, GARCH-type models, stochastic volatility models, jump-diffusion models, Levy-process models, more general semimartingale models ...

Problem:

Choose a pricing measure $\hat{Q}$ among all equivalent martingale measures

Some commonly used methods:

1. Parameterize $\hat{Q}_\theta$, $\theta \in \Theta$ and calibrate to market data of traded derivatives ... via $\frac{d\hat{Q}_\theta}{dP}$ or without $P$. E.g. build a stoch vol model or local vol model directly under $\hat{Q}$

2. Choose $\hat{Q}$ so that it minimizes some distance to $P$, e.g. $L^p$-distance, relative entropy, $f$-divergence ...

3. Indifference pricing

4. ...
Our goal: derive $\hat{\mathcal{Q}}$ from equilibrium considerations

Some motivating Examples

Horst and Müller (2007).
On the spanning property of risk bonds priced by equilibrium

Bakshi, Kapadia and Madan (2003).
Stock return characteristics, skew laws, and the differential pricing of individual equity options

Demand-based option pricing

Carmona, Fehr, Hinz and Porchet (2010).
Market design for emission trading schemes.
1. Model
2. Existence of equilibrium
3. Uniqueness of equilibrium
4. The case where agents can be aggregated
5. Random walks and BSΔEs
6. Brownian motion and BSDEs
Ingredients

- filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})\)
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- money market account with \(r \equiv 0\)
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- money market account with \(r \equiv 0\)
- exogenous asset \((R_t)^T_{t=0}\) satisfying (NA)
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- structured product in external supply \(n\) with final payoff \(S \in L^\infty(\mathcal{F}_T)\)
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- a group of finitely many agents \(\mathbb{A}\)
- agent \(a \in \mathbb{A}\) is endowed with an uncertain payoff \(H^a = g^{a,R} R_T + g^{a,S} S_T + G^a\)
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- structured product in external supply \(n\) with final payoff \(S \in L^\infty(\mathcal{F}_T)\)
- a group of finitely many agents \(\mathbb{A}\)
- agent \(a \in \mathbb{A}\) is endowed with an uncertain payoff \(H^a = g^{a,R} R_T + g^{a,S} S_T + G^a\)
- at time \(t\) agent \(a\) invests to optimize a preference functional

\[
U^a_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)
\]
We assume $U_t^a$ has the following properties:

(N) Normalization $U_t^a(0) = 0$
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(N) **Normalization**

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(M) **Monotonicity**

$U_t^a(X) \geq U_t^a(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$ such that $X \geq Y$

(T) **Translation property**

$U_t^a(X + Y) = U_t^a(X) + Y$ for all $X \in L^\infty(\mathcal{F}_T)$ and $Y \in L^\infty(\mathcal{F}_T)$

(TC) **Time-consistency**

$U_{t+1}^a(X) \geq U_{t+1}^a(Y) \implies U_t^a(X) = U_t^a(U_{t+1}^a(X))$
We assume $U^a_t$ has the following properties:

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$U^a_t(X) \geq U^a_t(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$ such that $X \geq Y$

(C) $\mathcal{F}_t$-Concavity
$U^a_t(\lambda X + (1 - \lambda)Y) \geq \lambda U^a_t(X) + (1 - \lambda)U^a_t(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$
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(TC) Time-consistency $U_{t+1}^a(X) \geq U_{t+1}^a(Y)$ implies $U_t^a(X) \geq U_t^a(Y)$

$\iff U_t^a(X) = U_t^a(U_{t+1}^a(X))$
Examples

1) \( U_t^a(X) = -\frac{1}{\gamma} \log \mathbb{E} [e^{-\gamma X} | \mathcal{F}_t] \)

2) \( U_t^a(X) = \mathbb{E} [X | \mathcal{F}_t] - \lambda \mathbb{E} [(X - \mathbb{E} [X | \mathcal{F}_t])^2 | \mathcal{F}_t] \)

3) \( U_t^a(X) = (1 - \lambda) \mathbb{E} [X | \mathcal{F}_t] - \lambda \rho_t(X) \)

where \( \rho_t \) is a conditional convex risk measure
An equilibrium of plans, prices and price expectations à la Radner (1972) consists of

- an adapted process \((S_t)_{t=0}^T\) with \(S_T = S\)
- trading strategies \((\hat{\vartheta}_t^a)_{t=1}^T\)

such that the following hold:

(i) individual optimality

\[
U^a_t \left( H^a + \sum_{s=t+1}^T \hat{\vartheta}_s^{a,1} \Delta R_s + \hat{\vartheta}_s^{a,2} \Delta S_s \right) 
\geq \quad U^a_t \left( H^a + \sum_{s=t+1}^T \vartheta_s^{a,1} \Delta R_s + \vartheta_s^{a,2} \Delta S_s \right)
\]

for every \(t\) and all possible strategies \((\vartheta_s^a)\)

(ii) market clearing \(\sum_{a \in A} \hat{\vartheta}_t^{a,2} = n\)
Hart (1975) On the optimality of equilibrium when the market structure is incomplete:

In general, a Radner equilibrium does not exist, and if there is one, it is not unique.
Existence of equilibrium

One-step representative agents

Set $H_T^a = H^a$ and

$$H_{t+1}^a = U_{t+1}^a \left( H^a + \sum_{s=t+2}^T \hat{\varphi}_{s,1}^{a,1} \Delta R_s + \hat{\varphi}_{s,2}^{a,2} \Delta S_s \right)$$

the correct one-step representative agent would be

$$\hat{u}_t(x) = \operatorname{ess sup}_{\varphi^a \in L^\infty(\mathcal{F}_t)^2} \sum_{a \in A} U_t^a \left( H_{t+1}^a + \varphi_{t+1,1}^{a,1} \Delta R_{t+1} + \varphi_{t+1,2}^{a,2} \Delta S_{t+1} \right)$$

$$\sum_{a \in A} \varphi^a, 2 = x$$
Existence of equilibrium

One-step representative agents

Set $H^a_T = H^a$ and

$$H^a_{t+1} = U^a_{t+1} \left( H^a + \sum_{s=t+2}^{T} \hat{\vartheta}_s^{a,1} \Delta R_s + \hat{\vartheta}_s^{a,2} \Delta S_s \right)$$

the correct one-step representative agent would be

$$\hat{u}_t(x) = \text{ess sup} \sum_{a \in A} U^a_t (H^a_{t+1} + \vartheta^{a,1} \Delta R_{t+1} + \vartheta^{a,2} \Delta S_{t+1})$$

$$\sum_{a \in A} \vartheta^{a,2} = x$$

But $S_t$ is not known. So define

$$\hat{u}_t(x) = \text{ess sup} \sum_{a} U^a_t (H^a_{t+1} + \vartheta^{a,1} \Delta R_{t+1} + \vartheta^{a,2} S_{t+1})$$

$$\sum_{a \in A} \vartheta^{a,2} = x$$

$\hat{u}_t$ is $\mathcal{F}_t$-concave
Existence of equilibrium

Convex dual characterization of equilibrium

**Theorem** A bounded, adapted process \((S_t)^T_{t=0}\) satisfying \(S_T = S\) together with trading strategies \((\hat{\vartheta}_t^a)^T_{t=1}, a \in A\), form an equilibrium \(\iff\) for all \(t\):

(i) \(S_t \in \partial \hat{u}_t(n)\)

(ii) \(\sum_{a \in A} U_t^a (H^a_{t+1} + \hat{\vartheta}_t^{a,1} \Delta R_{t+1} + \hat{\vartheta}_t^{a,2} S_{t+1}) = \hat{u}_t(n)\)

(iii) \(\sum_{a \in A} \hat{\vartheta}_t^{a,2} = n\)
Existence of equilibrium

Assumption (A)

For all $t = 0, \ldots, T - 1$, $V^a \in L^\infty(\mathcal{F}_{t+1})$, $W \in L^\infty(\mathcal{F}_{t+1})$, there exist $\hat{\vartheta}^a_{t+1} \in L^\infty(\mathcal{F}_t)^2$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}^{a,2}_{t+1} = 0$$

and

$$\sum_{a \in \mathbb{A}} U^a_t \left( V^a + \hat{\vartheta}^{a,1}_{t+1} \Delta R_{t+1} + \hat{\vartheta}^{a,2}_{t+1} W \right) = \text{ess sup} \sum_{a \in \mathbb{A}} U^a_t \left( V^a + \vartheta^{a,1}_{t+1} \Delta R_{t+1} + \vartheta^{a,2}_{t+1} W \right).$$

Lemma Under assumption (A) an equilibrium exists
Existence of equilibrium

**Definition**

$U_0^a$ is sensitive to large losses if

$$\lim_{\lambda \to \infty} U_0^a(\lambda X) = -\infty$$

for all $X \in L^\infty(\mathcal{F}_T)$ such that $\mathbb{P}[X < 0] > 0$.

**Theorem**

If all $U_0^a$ are sensitive to large losses, then condition (A) is satisfied and an equilibrium exists.

**Remark**

The theorem also works with convex trading constraints.
Existence of equilibrium

**Proposition**

If the market is in equilibrium and at least one agent has strictly monotone preferences and open trading constraints, then there exists a probability measure $Q \sim P$ such that $R_t = \mathbb{E}_Q [R_T | \mathcal{F}_t]$ and $S_t = \mathbb{E}_Q [S_T | \mathcal{F}_t]$. 
Differentiable preferences

We say $U_t^a$ satisfies the differentiability condition (D) if for all $X, Y \in L^\infty(\mathcal{F}_{t+1})$, there exists $Z \in L^1(\mathcal{F}_{t+1})$ such that

$$\lim_{k \to \infty} k \left( U_t^a \left( X + \frac{Y}{k} \right) - U_t^a(X) \right) = \mathbb{E} [YZ \mid \mathcal{F}_t].$$

If such a $Z$ exists, it has to be unique, and we denote it by $\nabla U_t^a(X)$. 
**Differentiable preferences**

We say $U^a_t$ satisfies the differentiability condition (D) if for all $X, Y \in L^\infty(\mathcal{F}_{t+1})$, there exists $Z \in L^1(\mathcal{F}_{t+1})$ such that

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$$

If such a $Z$ exists, it has to be unique, and we denote it by $\nabla U^a_t(X)$.

**Theorem** If at least one $U^a_t$ satisfies (D), then there can exist at most one equilibrium price process $(S_t)_{t=0}^{T}$, and if the market is in equilibrium, then

$$
\frac{dQ^a_t}{d\mathbb{P}} := \nabla U^a_t \left( H^a + \sum_{s=1}^{T} \hat{\vartheta}_s^{a,1} \Delta R_s + \hat{\vartheta}_s^{a,2} \Delta S_s \right)
$$

defines a pricing measure.
The case where agents can be aggregated

**One-Fund Theorem** Assume the following hold:

1. There is a unique equilibrium price process \((S_t)\)
2. The initial endowments are of the form

   \[ H^a = c^a + \sum_{t=1}^{T} \eta_{t,1}^a \Delta R_t + \sum_{t=1}^{T} \eta_{t,2}^a \Delta S_t \]

3. The preferences are of the form

   \[ U_t^a(X) = \frac{1}{\gamma^a} U_t(\gamma^a X) \]

   for \(\gamma^a > 0\) and a base preference \(U_t\)
The case where agents can be aggregated

Then agent a’s optimal one-step trading gains are

\[
\left( \frac{1}{\gamma^a} \hat{\vartheta}_t^1 - \eta_t^{a,1} \right) \Delta R_t + \left( \frac{\gamma}{\gamma^a} (n + \eta_t^2) - \eta_t^{a,2} \right) \Delta S_t,
\]

where

- \(1/\gamma := \sum_a 1/\gamma^a\)
- \(\eta_t^2 := \sum_a \eta_t^{a,2}\)
- \(\hat{\vartheta}_t^1\) is an optimal \(R\)-strategy for an agent with preferences \(U_t\) and investments in \(S\) of \(\gamma(n + \eta_t^2)\)

Moreover, if \(U_0\) is differentiable, then

\[
R_t = \mathbb{E}_Q[R_T | \mathcal{F}_t], \quad S_t = \mathbb{E}_Q[S_T | \mathcal{F}_t]
\]

for

\[
\frac{dQ}{dP} = \nabla U_0 \left( \sum_{t=1}^T \hat{\vartheta}_t^1 \Delta R_t + \gamma(n + \eta_t^2) \Delta S_t \right)
\]
Example

Effects of stochastic volatility, demand pressure and short-selling constraints on option prices
Example

Effects of stochastic volatility, demand pressure and short-selling constraints on option prices

- Agents = option dealers with expected exponential utility with parameter $\gamma^a > 0$
- Underlying = S&P 500

\[
R_{t+h} = |R_t + \mu R_t h + \sqrt{v_t} R_t \Delta b^1_{t+h}|
\]

\[
v_{t+h} = |v_t + \alpha (m - v_t) h + \beta \sqrt{v_t} \Delta b^2_{t+h}|
\]

- Pension funds demand put options $(K - R_T)^+$
Zero endowments and stochastic volatility

(I) No demand pressure

frequent hedging  infrequent hedging
Equilibrium prices of out-of-the-money put options

Zero endowments and stochastic volatility

(II) Positive demand pressure

frequent hedging

infrequent hedging
Zero endowments and stochastic volatility

(III) Positive demand pressure and short selling constraints
Random walks and BSΔEs

Fix $h > 0$ and $N \in \mathbb{N}$

Denote $T = \{0, h, \ldots, T = Nh\}$

$b^1_t, \ldots, b^d_t$ $d$ independent random walks with

$P[\Delta b^i_{t+h} = \pm \sqrt{h}] = 1/2$

$b^{d+1}_t, \ldots, b^D_t$ $2^d - (d + 1)$ random walks orthogonal to $b^1_t, \ldots, b^d_t$

Every $X \in L^\infty(\mathcal{F}_{t+h})$ can be represented as

$$X = \mathbb{E}[X | \mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+h}$$

for

$$\pi_t(X) \cdot \Delta b_{t+h} = \sum_{i=1}^{D} \pi^i_t(X) \Delta b^i_{t+h} \quad \text{and} \quad \pi^i_t(X) = \frac{1}{h} \mathbb{E}[X \Delta b^i_{t+h} | \mathcal{F}_t].$$
\[ U_t^a(X) = U_t^a (\mathbb{E} [X | \mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+h}) = \mathbb{E} [X | \mathcal{F}_t] - f_t^a(\pi_t(X))h \]

for the \( \mathcal{F}_t \)-convex function \( f_t^a : L^\infty(\mathcal{F}_t)^D \to L^\infty(\mathcal{F}_t) \) given by

\[ f_t^a(z) := -\frac{1}{h} U_t^a(z \cdot \Delta b_{t+h}). \]

Assume condition (A) is satisfied and all \( U_t^a \) satisfy the differentiability condition (D).

Then there exists \( \nabla f_t^a(z) \in L^\infty(\mathcal{F}_t)^D \) such that

\[ \lim_{k \to \infty} k \left( f_t^a(z + z'/k) - f_t^a(z) \right) = z' \cdot \nabla f_t^a(z) \]
Random walks and BS∆Es

For given $R_{t+h}, S_{t+h}, H^a_{t+h}$ denote

\[
Z^{R}_{t+h} := \pi_t(R_{t+h}) \\
Z^{S}_{t+h} := \pi_t(S_{t+h}) \\
Z^{a}_{t+h} := \pi_t(H^a_{t+h}) \\
Z_{t+h} = (Z^{R}_{t+h}, Z^{S}_{t+h}, Z^{a}_{t+h}, a \in A).
\]

and define the function $f_t : L^\infty(\mathcal{F}_t)^{\left(3+|A|\right)D} \to L^\infty(\mathcal{F}_t)$ by

\[
f_t(v, Z_{t+h}) = \text{ess inf} \sum_{a \in A} \sum_{\vartheta^a \in L(\mathcal{F}_t)^2} f^a_t \left( \frac{v}{|A|} + Z^a_{t+h} + \vartheta^a_{t+h} Z^{R}_{t+h} + \vartheta^a_{t+h} Z^{S}_{t+h} \right)
\]

\[
-\vartheta^a_{t+h} \frac{\mathbb{E} [\Delta R_{t+h} | \mathcal{F}_t]}{h}.
\]
Set

\[ g^S_t(Z_{t+h}) := Z^S_{t+h} \cdot \nabla^v f_t (n Z^S_{t+h}, Z_{t+h}) \]
\[ g^a_t(Z_{t+h}) := f^a_t \left( Z^a_{t+h} + \hat{\nu}^{a,1}_{t+h} Z^R_{t+h} + \hat{\nu}^{a,2}_{t+h} Z^S_{t+h} \right) \]
\[ -\hat{\nu}^{a,1}_{t+h} \frac{1}{h} \mathbb{E} [\Delta R_{t+h} | \mathcal{F}_t] - \hat{\nu}^{a,2}_{t+h} g^S_t (Z_{t+h}). \]
The processes \((S_t)\) and \((H^a_t)\) satisfy the following coupled system of BS\&Es

\[
\Delta S_{t+h} = g_t^S(Z_{t+h})h + Z_{t+h}^S \cdot \Delta b_{t+h}, \quad S_T = S
\]
\[
\Delta H^a_{t+h} = g_t^a(Z_{t+h})h + Z_{t+h}^a \cdot \Delta b_{t+h}, \quad H^a_T = H.
\]
Example

Assume that the price of the exogenous asset is given by

$$\Delta R_{t+h} = R_t(\mu h + \sigma \Delta b^1_{t+h}) , \quad R_0 > 0$$

and agent a’s preference functional is

$$U^a_t(X) = -\frac{1}{\gamma^a} \log \mathbb{E}[\exp(-\gamma^a X) | \mathcal{F}_t] \quad \text{for some} \quad \gamma^a > 0 .$$

Then

$$U^a_t(X) = \mathbb{E}[X | \mathcal{F}_t] - f^a_t(\pi_t(X))h$$

for

$$f^a_t(z) = \frac{1}{h\gamma^a} \log \mathbb{E}[\exp(-\gamma^a z \cdot \Delta b_{t+h})] .$$
Neglect the random walks $b^{d+1}, \ldots, b^D$

and use the approximation

$$\frac{1}{\hbar \gamma^a} \sum_{i=1}^{d} \log \cosh \left( \sqrt{\hbar \gamma^a} z^i \right) \approx \frac{\gamma^a}{2} \sum_{i=1}^{d} (z^i)^2$$

Then the BS$\Delta$E of the last theorem yields ...
Random walks and BS∆Es

... the recursive algorithm

\[
S_t = \mathbb{E}[S_{t+1} | \mathcal{F}_t] - g_t^S h, \quad S_T = S
\]

\[
H_t^a = \mathbb{E}[H_{t+1}^a | \mathcal{F}_t] - g_t^a h, \quad H_T^a = H^a,
\]

where

\[
g_t^S = \frac{1}{c_{RR}} \left[ c_{RS} \mu S_t + \gamma \left( n \left\{ c_{RR} c_{SS} - c_{RS} c_{RS} \right\} + c_{RA} c_{RR} - c_{SR} c_{RA} \right) \right]
\]

\[
g_t^a = \frac{\gamma^a}{2} \left\| Z_{t+h}^a + \hat{\jmath}_{t+h}^{a,1} Z_{t+h}^R + \hat{\jmath}_{t+h}^{a,2} Z_{t+h}^S \right\|^2 - \hat{\jmath}_{t+h}^{a,1} \mu R_t - \hat{\jmath}_{t+h}^{a,2} G_t^S
\]

\[
\hat{\jmath}_{t+h}^{a,1} = \frac{\mu S_t}{\gamma^a c_{RR}} + \frac{c_{SR} c_{Sa} - c_{Ra} c_{SS}}{c_{RR} c_{SS} - c_{RS} c_{RS}} - \frac{c_{RS}}{c_{RR} \gamma^a} \left( n + \frac{c_{RR} c_{SA} - c_{RS} c_{AR}}{c_{RR} c_{SS} - c_{RS} c_{RS}} \right)
\]

\[
\hat{\jmath}_{t+h}^{a,2} = n \frac{\gamma}{\gamma^a} + \frac{c_{RS} c_{Ra} - c_{Sa} c_{RR} - \frac{\gamma}{\gamma^a} \left( c_{RS} c_{RA} - c_{RR} c_{SA} \right)}{c_{RR} c_{SS} - c_{RS} c_{RS}}
\]

for \( \gamma := (\sum_a (\gamma^a)^{-1})^{-1} \),

\[
c_{RR} := Z_{t+h}^S \cdot Z_{t+h}^S, \quad c_{SR} := Z_{t+h}^S \cdot Z_{t+h}^R, \quad c_{SA} := Z_{t+h}^S \cdot \sum_a Z_{t+h}^a,
\]
Example

Let $B^R_t, B^S_t, B^a_t, a \in \mathbb{A}$, be independent Brownian motions

$$dR_t = \mu R_t dt + \sigma R_t dB^R_t, \quad R_0 > 0$$

and suppose agent a’s preference functional is

$$U^a_t(X) = -\frac{1}{\gamma^a} \log \mathbb{E}[\exp(-\gamma^a X) \mid \mathcal{F}_t] \quad \text{for some} \quad \gamma^a > 0.$$
Brownian motion and BSDEs

The BSDE corresponding to the above BSDE is

\[ dS_t = g_t^S dt + Z_t^S \cdot dB_t, \quad S_T = S \]
\[ dH_t^a = g_t^a dt + Z_t^a \cdot dB_t, \quad H_T^a = H^a, \]

where

\[
\begin{align*}
g_t^S &= \frac{1}{c_{RR}} \left[ c_{RS} \mu S_t + \gamma \left( n \left\{ c_{RR} c_{SS} - c_{RS} c_{RS} \right\} + c_{RA} c_{RR} - c_{SR} c_{RA} \right) \right] \\
g_t^a &= \frac{\gamma^a}{2} \left\| Z_t^a + \hat{\gamma}^{a,1}_t Z_t^R + \hat{\gamma}^{a,2}_t Z_t^S \right\|^2 - \hat{\gamma}^{a,1}_t \mu R_t - \hat{\gamma}^{a,2}_t g_t^S \\
\hat{\gamma}^{a,1}_t &= \frac{\mu S_t}{\gamma^a c_{RR}} + \frac{c_{SR} c_{Sa} - c_{Ra} c_{SS}}{c_{RR} c_{SS} - c_{RS} c_{RS}} - \frac{c_{RS}}{c_{RR}} \frac{\gamma}{\gamma^a} \left( n + \frac{c_{RR} c_{SA} - c_{RS} c_{AR}}{c_{RR} c_{SS} - c_{RS} c_{RS}} \right) \\
\hat{\gamma}^{a,2}_t &= n \frac{\gamma}{\gamma^a} + \frac{c_{RS} c_{Ra} - c_{Sa} c_{RR} - \gamma^a \left( c_{RS} c_{RA} - c_{RR} c_{SA} \right)}{c_{RR} c_{SS} - c_{RS} c_{RS}} \\
\end{align*}
\]

for

\[
c_{RR} := Z_t^R \cdot Z_t^R, \quad c_{RS} := Z_t^R \cdot Z_t^S, \quad c_{RA} := Z_t^R \cdot \sum_a Z_t^a, \quad ...\]