Dynamic Conic Finance
Pricing and Hedging via Dynamic Coherent Acceptability Indices with Transaction Costs

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Bibliography


- Tomasz R. Bielecki, Ig. Cialenco and Zhao Zhang, Dynamic Coherent Acceptability Indices and their Applications to Finance, Forthcoming in Mathematical Finance
Outline of the talk

- Introduction and overview of the existing methods

- Main Ingredients:
  - Dynamic Coherent Acceptability Indices
  - Dynamic Coherent Risk Measures

- No-Good-Deals and Fundamental Theorem of Assets Pricing

- Good-deal ask and bid prices

- Examples
Incomplete Financial Market Models

- not every contingent claim (financial contract) is hedgeable (or attainable, replicable, marketable)

- there are infinitely many risk-neutral probability measures (assuming no arbitrage)

- fair price = $E^Q[\text{Payoff}^*]$, where $Q$ is a risk-neutral probability;

- if the claim is not hedgeable, then fair price of a contingent claim is not unique

- Nevertheless one can compute no arbitrage bounds for prices

\[
\left[ \inf_{Q \in \mathcal{R}} E^Q[\text{Payoff}^*], \sup_{Q \in \mathcal{R}} E^Q[\text{Payoff}^*] \right]
\]

where $\mathcal{R}$ is the set of all risk neutral probabilities
Introduction

- **No arbitrage bounds** for prices

\[
\left[ \inf_{Q \in \mathcal{R}} \mathbb{E}^Q[\text{Payoff}^*], \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[\text{Payoff}^*] \right]
\]

where \( \mathcal{R} \) is the set of all risk neutral probabilities

- However, the **no arbitrage bounds are too wide** in practice

- Shrinking the arbitrage free price interval:

  (a) Indifference pricing via utility maximization - a price at which an agent receives the same expected utility between trading and not trading (book by Carmona '09). Limitations: numerical implementations and explicit calculations may not be robust; resulting bid and ask prices are not necessarily risk-neutral in practice (Staum '07)

  (b) Rule out deals that are **too good to be true**, eliminating prices with high Sharpe ratios (Cochrane and Requejo '99)
Methodology

\[
\inf_{Q \in \mathcal{R}} \mathbb{E}^Q[X^*] \leq \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X^*] \quad \text{no arbitrage}
\]
Similar to arbitrage opportunities, **good-deals** should not be available in the market, since everyone would be willing to take them.

- A market maker wants to determine the best bid and ask prices to offer to the market for a specified **acceptability level** that he/she picks.

- Easy to compute, robust, does not contradict the general arbitrage theory.
Bernardo and Ledoit ’00 and Pinar, Salih, and Camci ’10 - cash flows are good deals if the Gain-Loss Ratio is high

Good-deal-bound approach has been generalized and used in applications by Carr et al. ’01, Jaschke and Kuchler ’01, Staum ’04, Engwerda et al. ’05, Bjork and Slinko ’06, Kloppel and Schweitzer ’07, Arai and Fukasawa ’11.

Using coherent risk measures Cherny and Madan ’06, ’07

Dynamic bid and ask prices via dynamic risk measures

**Conic Finance** - static bid and ask prices by *Acceptability Indices* - Madan and Cherny ’11;
no bid/ask allowed for hedgeable securities
Our contribution

- Extend Good-Deal bound setup using **Dynamic Coherent Acceptability Indices** and **Dynamic Coherent Risk Measures** (Bielecki, Cialenco and Zhang ’11)

- Price (bid/ask) contingent claims that pay dividends (CDS, IRS); the input is a process

- Allow transaction cost for underlying/headgeable securities
Our contribution

- Extend Good-Deal bound setup using **Dynamic Coherent Acceptability Indices** and **Dynamic Coherent Risk Measures** (Bielecki, Cialenco and Zhang ’11)

- Price (bid/ask) contingent claims that pay dividends (CDS, IRS); the input is a process

- Allow transaction cost for underlying/ hedgeable securities

**Dynamic Conic Finance**
Keyword - **DYNAMIC**
Motivation and Preliminaries

Measures of Performance

- **Sharpe Ratio**

\[
SR(X) = \frac{\mathbb{E}[X - r]}{\text{Std}[X]}
\]

- **Gain-Loss Ratio**

\[
\text{GLR}(X) := \begin{cases} 
\frac{\mathbb{E}[X]}{\mathbb{E}[X^-]}, & \text{if } \mathbb{E}[X] > 0 \\
0, & \text{otherwise}
\end{cases}
\]

- **Risk Adjusted Return on Capital** \( RAROC(X) = \frac{\mathbb{E}(X)}{\rho(X)} \)

- **Treynor Ratios** \( TR(X) = \frac{\mathbb{E}(X) - r}{\beta(X)} \)

- **Tilt Coefficient** \( TC(X) = \sup\{\lambda \in \mathbb{R}^+ \mid \mathbb{E}[X e^{-\lambda X}] \geq 0\} \)

- and more AIMINMAX, AIMAXMIN
Motivation and Preliminaries

Measures of Risk

- **Value at Risk**
  \[
  \text{VaR}_\alpha(X) = \inf \{ m \in \mathbb{R} \mid \mathbb{P}[X + m < 0] \leq \alpha \} = -q_\alpha(X)
  \]

- **Average VaR or Tail Value at Risk**
  \[
  \text{AVaR} = \frac{1}{\alpha} \int_{0}^{\alpha} \text{VaR}_\beta(X) d\beta
  \]

- **Expected Shortfall or Tail Conditional Expectation**
  \[
  \text{ES}(X) = \mathbb{E}[-X \mid -X \geq \text{VaR}(X)]
  \]

- **Entropic Risk Measure**
  \[
  \text{Ent}_\lambda(X) = \frac{1}{\lambda} \log \mathbb{E}[\exp(-\lambda X)]
  \]
Notations, General Assumptions

- $T$-fixed time horizon, $\mathcal{T} := \{0, 1, \ldots, T\}$

- $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{P})$ - the underlying filtered probability space, with $\Omega$ finite

- $L^0 := L^0(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$ - the set of all $\mathbb{F}$-adapted processes

- $\{B_t\}_{t \in \mathcal{T}}$ - banking account

- Any $D \in L^0$ is interpreted as a dividend stream

- $D^* := \frac{D}{B}$ - discounted dividend stream
Dynamic Coherent Acceptability Index (DCAI):

\( \alpha : \mathcal{T} \times L^0 \times \Omega \to [0, +\infty] \) that satisfies the following properties:

1. **Monotonicity.** If \( D_s \geq D'_s \) for all \( s \geq t \), then \( \alpha_t(D) \geq \alpha_t(D') \)

2. **Scale invariance.** \( \alpha_t(\lambda D) = \alpha_t(D) \) for all \( \lambda > 0 \)

3. **Quasi-concavity.** If \( \alpha_t(D, \omega) \geq x \) and \( \alpha_t(D', \omega) \geq x \), then \( \alpha_t(\lambda D + (1 - \lambda)D', \omega) \geq x \) for all \( \lambda \in [0, 1] \)

4. **Adaptiveness.** \( \alpha_t(D) \) is \( \mathcal{F}_t \)-measurable

5. **Independence of the past.** If \( 1_A D_s = 1_A D'_s \) for \( A \in \mathcal{F}_t \) and for all \( s \geq t \), then \( 1_A \alpha_t(D) = 1_A \alpha_t(D') \)

6. **Translation invariance.** \( \alpha_t(D + m1_{\{t\}}) = \alpha_t(D + m1_{\{s\}} \frac{B_s}{B_t}) \), \( s \geq t \)

7. **Dynamic consistency.** Let \( D \in \mathcal{D} \), and \( m \geq 0 \) be \( \mathcal{F}_t \) measurable
   
   (a) If \( D_t \geq 0 \) and \( \alpha_{t+1}(D) \geq m \), then \( \alpha_t(D) \geq m \)
   
   (b) If \( D_t \leq 0 \) and \( \alpha_{t+1}(D) \leq m \), then \( \alpha_t(D) \leq m \)
Dynamic Coherent Acceptability Index (DCAI): Discussion

- **Monotone, Quasi-concave, Unitless**

- Generalization of Sharpe Ratio $SR(X) = \mathbb{E}[X - r]/\text{Std}[X]$
  
  SR is not monotone

- **Main Example:**
  
  **Dynamic Gain-Loss Ratio**

  \[
  \text{dGLR}_t(D)(\omega) := \begin{cases} 
  \frac{\mathbb{E}_t^P \left[ \sum_{s=t}^T D_s^* \right](\omega)}{\mathbb{E}_t^P \left[ \left( \sum_{s=t}^T D_s^* \right)^- \right](\omega)}, & \text{if } \mathbb{E}_t^P \left[ \sum_{s=t}^T D_s^* \right](\omega) > 0 \\
  0, & \text{otherwise}
  \end{cases}
  \]

  for all $t \in \mathcal{T}$, and $\omega \in \Omega$.

  Static $\text{GLR}(X) = \mathbb{E}[X]/\mathbb{E}[X^-]$
Dynamic Coherent Risk Measure (DCRM):

\[ \rho : \mathcal{T} \times L^0 \times \Omega \to \mathbb{R} \] that satisfies the following properties:

1. **Monotonicity.** If \( D_s \geq D'_s \) for all \( s \geq t \), then \( \rho_t(D) \leq \rho_t(D') \)

2. **Homogeneity.** \( \rho_t(\lambda D) = \lambda \rho_t(D) \) for all \( \lambda > 0 \)

3. **Subadditivity.** \( \rho_t(D + D') \leq \rho_t(D) + \rho_t(D') \)

4. **Adaptiveness.** \( \rho_t(D) \) is \( \mathcal{F}_t \)-measurable

5. **Independence of the past.** If \( 1_A D_s = 1_A D'_s \) for \( A \in \mathcal{F}_t \) and for all \( s \geq t \), then \( 1_A \rho_t(D) = 1_A \rho_t(D') \)

6. **Translation invariance.** \( \rho_t(D + m \{ s \} \frac{B_s}{B_t}) = \rho_t(D) - m, \ s \geq t \)

7. **Dynamic consistency.** For all \( A \in \mathcal{F}_t, D \in L^0 \)

\[ 1_A(\min_{\omega \in A} \rho_{t+1}(D, \omega) - D_t) \leq 1_A \rho_t(D) \leq 1_A(\max_{\omega \in A} \rho_{t+1}(D, \omega) - D_t) \]

Generalization of Value At Risk \( \text{V@R} \); Measured in monetary units (\$)
Duality and Robust Representations for DCAIs

\[ \alpha \leftrightarrow \{ \rho^\gamma \}_{\gamma \in \mathbb{R}^+} \]

DAI \leftrightarrow increasing family of DCRMs

\[ \alpha_t(D)(\omega) = \sup\{ \gamma \in (0, \infty) : \rho_t^\gamma(D)(\omega) \leq 0 \} \]
Theorem (Bielecki, C., Zhang, 2011)

- If $\alpha$ is a normalized, right-continuous DCAI then there exists a left-continuous and increasing family of DCRMs such that

$$
\alpha_t(D)(\omega) = \sup\{\gamma \in (0, \infty) : \rho_t^\gamma(D)(\omega) \leq 0\},
$$

for all $\omega \in \Omega$, $t \in \mathcal{T}$, $D \in L^0$.

- If $(\rho^\gamma)_{\gamma \in (0,\infty)}$ is a left-continuous and increasing family of DCRMs, then there exists a right-continuous and normalized DCAI $\alpha$ such that

$$
\rho_t^\gamma(D)(\omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + \delta_t^*(1c))(\omega) \geq \gamma\},
$$

for all $\omega \in \Omega$, $t \in \mathcal{T}$, $D \in L^0$. 

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Duality and Representation Theorems

Duality for Risk Measures

For a fixed $\gamma \in \mathbb{R}_+$

$$\rho^\gamma \longleftrightarrow \{Q_t^\gamma\}_{t=0}^T$$ a set of probability measures

$Q^\gamma = \{Q_t^\gamma\}_{t=0}^T$ increasing, dynamic consistent sets of probabilities

Theorem (Robust Representation of DCRMs)

A function $\rho^\gamma$ is a DCRM if and only if there exists $(Q_t^\gamma)_{t=0}^T$ such that,

$$\rho_t^\gamma(D) = -\inf_{Q \in Q_t^\gamma} \mathbb{E}_t^Q \left[ \sum_{s=t}^T D_s^* \right]$$

Theorem (Robust Representation of DCAIs)

$\alpha$ is a DCAI if and only if there exists $(Q_t^\gamma)_{t=0}^T$ such that

$$\alpha_t(D)(\omega) = \sup \left\{ \gamma \in (0, \infty) : \inf_{Q \in Q_t^\gamma} \mathbb{E}_t^Q \left[ \sum_{s \geq t} D_s^* \right](\omega) \geq 0 \right\}$$
Dynamically Consistent Sets of Probability Measures

**Definition**

(i) A sequence of sets of probability measures \((Q_t)_{t=0}^T\) absolutely continuous with respect to \(\mathbb{P}\) is called **dynamically consistent with respect to the filtration** \((\mathcal{F}_t)_{t=0}^T\) if the sequence is of full-support and the following inequality holds true

\[
\mathbb{I}_{E} \min_{\omega \in E} \left\{ \inf_{Q \in Q_{t+1}} \mathbb{E}_t^Q [X](\omega) \right\} \leq \mathbb{I}_{E} \inf_{Q \in Q_t} \mathbb{E}_t^Q [X]
\]

\[
\leq \mathbb{I}_{E} \max_{\omega \in E} \left\{ \inf_{Q \in Q_{t+1}} \mathbb{E}_t^Q [X](\omega) \right\}
\]

for all \(t \in \{0, 1, \ldots, T-1\}\), \(E \in \mathcal{F}_t\), and \(\mathcal{F}_T\)-measurable random variables \(X\).

(ii) A family of sequences of sets of probability measures \(((Q_t^\gamma)_{t=0}^T)_{\gamma \in (0,\infty)}\) is called **increasing** if \(Q_t^\gamma \supseteq Q_t^\beta\), for all \(\gamma \geq \beta > 0\) and \(t \in T\).
Arbitrage and Good-deals

Financial Market Model: underlying assets

- \( B := \left( \prod_{s=0}^{t} (1 + r_s) \right)_{t=0}^{T} \) is the savings account.

- \( P^{\text{ask}}_{t} := \left( (P^{\text{ask},1}_t, \ldots, P^{\text{ask},N}_t) \right)_{t=0}^{T} \) is the ex-dividend price process;
  \( A^{\text{ask}}_{t} := \left( (A^{\text{ask},1}_t, \ldots, A^{\text{ask},N}_t) \right)_{t=1}^{T} \) is the associated (cumulative) dividend process.

- \( P^{\text{bid}}_{t} := \left( (P^{\text{bid},1}_t, \ldots, P^{\text{bid},N}_t) \right)_{t=0}^{T} \) is the ex-dividend price process;
  \( A^{\text{bid}}_{t} := \left( (A^{\text{bid},1}_t, \ldots, A^{\text{bid},N}_t) \right)_{t=1}^{T} \) is the cumulative dividend process.

- **Natural Assumption:** \( P^{\text{ask}}_{t} \geq P^{\text{bid}}_{t}, \Delta A^{\text{bid}}_{t} \geq \Delta A^{\text{ask}}_{t} \).

Goals

- build up an arbitrage free theory for this market
- use DCAI, DCRM and good-deal bounds to produce consistent bid/ask prices for contingent claims in this market
Example: Credit Default Swap (CDS) contract

- $\tau$ be the random time of the credit event of the reference entity
- Initiation date $t = 0$, maturity $t = T$, nominal value of $\$1$, and the loss-given-default $\delta \geq 0$ paid at default
- $\kappa^{bid}$ is the CDS spread quoted by the dealer to sell protection
- $\kappa^{ask}$ is the CDS spread quoted by the dealer to buy protection
- For the CDS contract specified above,

$$A_t^{ask} := 1_{\{\tau \leq t\}} \delta - \kappa^{ask} \sum_{u=1}^{t} 1_{\{u<\tau\}},$$

$$A_t^{bid} := 1_{\{\tau \leq t\}} \delta - \kappa^{bid} \sum_{u=1}^{t} 1_{\{u<\tau\}}, \quad t \in T$$

- $P_t^{ask}$ and $P_t^{bid}$ are the mark-to-market prices of the CDS
The value process $V(\phi)$ associated with a trading strategy $\phi$ is defined as

$$V_t(\phi) = \begin{cases} \phi_0^t + \sum_{j=1}^{N} 1\{\phi_t^j \geq 0\} \phi_t^j P_0^{ask,j} + \sum_{j=1}^{N} 1\{\phi_t^j < 0\} \phi_t^j P_0^{bid,j}, & \text{if } t = 0, \\ \phi_t^0 B_t + \sum_{j=1}^{N} 1\{\phi_t^j \geq 0\} \phi_t^j (P_t^{bid,j} + \Delta A_t^{ask,j}) \\ + \sum_{j=1}^{N} 1\{\phi_t^j < 0\} \phi_t^j (P_t^{ask,j} + \Delta A_t^{bid,j}), & \text{if } 1 \leq t \leq T. \end{cases}$$

A trading strategy $\phi$ is self-financing if for all $t = 1, 2, \ldots, T - 1$

$$B_t \Delta \phi_t^0 + \sum_{j=1}^{N} P_t^{ask,j} 1\{\Delta \phi_t^{j+1} \geq 0\} \Delta \phi_t^{j+1} + \sum_{j=1}^{N} P_t^{bid,j} 1\{\Delta \phi_t^{j+1} < 0\} \Delta \phi_t^{j+1}$$

$$= \sum_{j=1}^{N} \phi_t^j 1\{\phi_t^j \geq 0\} \Delta A_t^{ask,j} + \sum_{j=1}^{N} \phi_t^j 1\{\phi_t^j < 0\} \Delta A_t^{bid,j}.$$
Hedging cash flows at zero cost

- Set of self-financing trading strategies initiated at time $t$:
  $$S(t) := \begin{cases} \{ \phi : \phi \text{ is s.f., } V_0(\phi) = 0 \}, & t = 0 \\ \{ \phi : \phi \text{ is s.f., } \phi_s = 1_{\{s \geq t+1\}} \phi_s, \ s = 1, 2, \ldots, T \}, & t = 1 \ldots, T - 1 \end{cases}$$

Any $\phi \in S(t)$ is of the form $(0, \ldots, 0, \phi_{t+1}, \phi_{t+2}, \ldots, \phi_T)$

- Set of hedging cash flows initiated at time $t$:
  $$\mathcal{H}^0(t) := \left\{ \left(0, \ldots, 0, \Delta V^*_t(\phi), \ldots, \Delta V^*_T(\phi) \right) : \phi \in S(t) \right\}$$
  for $t \in \{0, \ldots, T - 1\}$, where $V^*(\phi)$ is the discounted wealth process, and $\Delta V^*_t = V^*_{t+1} - V^*_t$

- Note that for any $H \in \mathcal{H}^0(t)$
  $$\sum_{s=t+1}^{T} H_s = V^*_T(\phi), \quad H_t = 0.$$
No-arbitrage condition and risk-neutral measures

Definition

An arbitrage opportunity at time \( t \in \{0, \ldots, T-1\} \) for \( \mathcal{H}^0(t) \) is a cash flow \( H \in \mathcal{H}^0(t) \) such that \( \sum_{s=t}^{T} H_s(\omega) \geq 0 \) for all \( \omega \in \Omega \), and \( \mathbb{E}_{P}^{t}[\sum_{s=t}^{T} H_s](\omega) > 0 \) for some \( \omega \in \Omega \).

A probability measure \( Q \) is risk-neutral for \( \mathcal{H}^0(t) \) if \( Q \sim P \), and if \( \mathbb{E}_{Q}^{t}[\sum_{s=t}^{T} H_s](\omega) \leq 0 \) for all \( \omega \in \Omega \) and all \( H \in \mathcal{H}^0(t) \).

- \( \mathcal{R}(\mathcal{H}^0(t)) \) is the set of all risk-neutral measures
- NA has the usual interpretation of “not making something out of nothing”
- NA and Risk-Neutral agree with classical theory
No-good-deal condition

Definition

The **No-Good-Deal (NGD)** condition holds true for $\mathcal{H}(t)$ at time $t \in \{0, \ldots, T - 1\}$ and level $\gamma > 0$ if $\rho_t^\gamma(H)(\omega) \geq 0$ for all $H \in \mathcal{H}(t)$ and $\omega \in \Omega$.

- $\{\rho^\gamma\}_{\gamma \in \mathbb{R}_+}$ - a family of DCRMs
- The no-good-deal condition for different times are related through the dynamical consistency property of $\rho^\gamma$
- If NGD is satisfied for $\gamma > 0$, then it is also satisfied for $\gamma' > \gamma$ since $\rho^\gamma$ is increasing in $\gamma$
- NGD depends on the choice of $\rho^\gamma$
Fundamental theorem of NGD pricing

Theorem

NGD is satisfied for $\mathcal{H}(t)$ at time $t \in \{0, \ldots, T - 1\}$ and level $\gamma > 0$ if and only if $\mathcal{R}(\mathcal{H}(t)) \cap Q^\gamma_t \neq \emptyset$.

- If NGD is satisfied, then NA is also satisfied.
- Necessity is an immediate consequence of the robust representation theorem for DCRMs.
- Sufficiency is more involved. Requires a separation argument and the duality theorem between DCAIs and DCRMs.
Good-deal ask and bid prices

Let $\delta^+_t, \delta_t : L^0 \to L^0$ as follows

$$
\delta^+_t(D) := (0, \ldots, 0, D_{t+1}, \ldots, D_T), \quad t \in \{0, \ldots, T-1\}
$$

$$
\delta_t(D) := (0, \ldots, 0, D_t, 0, \ldots, 0), \quad t \in T
$$

**Definition**

The discounted *good-deal ask and bid prices* of a derivative contract $D$, at level $\gamma > 0$, at time $t \in \{1, \ldots, T-1\}$ are defined as

$$
\Pi_{t}^{\text{ask}, \gamma}(D)(\omega) := \inf\{v \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(1v) + H - \delta^+_t(D^*))(|\omega|) \geq \gamma\}
$$

$$
\Pi_{t}^{\text{bid}, \gamma}(D)(\omega) := \sup\{v \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta^+_t(D^*) + H - \delta_t(1v))(\omega) \geq \gamma\}
$$

- Prices at different times are related by the dynamic consistency property of $\alpha$
- We have that $\Pi_t^{\text{ask}, \gamma}(D) = -\Pi_t^{\text{bid}, \gamma}(-D)$
- Prices depend on $\alpha$, level $\gamma$, and hedging cash flows $\mathcal{H}(t)$
The discounted good-deal ask and bid prices of a derivative contract $D \in L^0$, at level $\gamma > 0$, at time $t \in \{1, \ldots, T - 1\}$ satisfy

$$
\Pi_t^{ask, \gamma}(D) = \sup_{Q \in Q_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^Q \left[ \sum_{s=t+1}^{T} D_s^* \right]
$$

$$
\Pi_t^{bid, \gamma}(D) = \inf_{Q \in Q_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^Q \left[ \sum_{s=t+1}^{T} D_s^* \right]
$$

- $\Pi_t^{ask, \gamma}(D)$ and $\Pi_t^{bid, \gamma}(D)$ are risk-neutral
- Since $Q_t^\gamma$ is $\uparrow$ in $\gamma$, we have that $\Pi_t^{ask, \gamma}(D) - \Pi_t^{bid, \gamma}(D)$ is $\uparrow$ in $\gamma$
- $\Pi_t^{ask, \gamma}(D) \geq \Pi_t^{bid, \gamma}(D)$
Discounted superhedging ask and bid prices are given as

\[ S_{0}^{bid}(D) = \inf_{Q \in \mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{Q}[\sum_{s=1}^{Y} D_{s}^*] \]

\[ S_{0}^{ask}(D) = \sup_{Q \in \mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{Q}[\sum_{s=1}^{Y} D_{s}^*] \]

**Shrinking superhedging price interval**

\[ S_{0}^{bid}(D) \leq \Pi_{0}^{bid,\gamma}(D) \leq \Pi_{0}^{ask,\gamma}(D) \leq S_{0}^{ask}(D) \]
Assume that the risk-free rate is deterministic.

**Definition**

The *good-deal ask and bid forward prices*, with delivery at time $T$, written at time $t \in \{1, \ldots, T - 1\}$, of a derivative contract $D \in L^0$, at level $\gamma > 0$ are defined as

$$
F_{t}^{\text{ask}, \gamma, T}(D)(\omega) := \inf\{f \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t.} \alpha_t(\delta_T(1B_T^{-1}f) + H - \delta_t^+(D^*)))(\omega) \geq \gamma\},
$$

$$
F_{t}^{\text{bid}, \gamma, T}(D)(\omega) := \sup\{f \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t.} \alpha_t(-\delta_t(1B_T^{-1}f) + H + \delta_t^+(D^*)))(\omega) \geq \gamma\}
$$

for all $\omega \in \Omega$.

- The case in which $r$ is random is much harder
Theorem

The good-deal ask and bid forward prices of a derivative contract $D \in L^0$, with delivery at time $T$, written at time $t \in \{1, \ldots, T - 1\}$ and level $\gamma > 0$, satisfy

$$F^\text{ask},\gamma,T_t(D)(\omega) = B_T \Pi^\text{ask},\gamma_t(D),$$

$$F^\text{bid},\gamma,T_t(D)(\omega) = B_T \Pi^\text{bid},\gamma_t(D).$$
Definition

\[ \text{dGLR}_t(D)(\omega) := \begin{cases} 
\frac{\mathbb{E}^P_t[\sum_{s=t}^T D_s^*](\omega)}{\mathbb{E}^P_t\left[\left(\sum_{s=t}^T D_s^*\right)^-\right](\omega)}, & \text{if } \mathbb{E}^P_t\left[\sum_{s=t}^T D_s^*\right](\omega) > 0, \\
0, & \text{otherwise},
\end{cases} \]

for all \( t \in \mathcal{T} \), and \( \omega \in \Omega \).

- It is shown in Bielecki, Cialenco, Zhang 2011 that the dGLR is a dynamic coherent acceptability index.
- NA is satisfied at time \( t \in \mathcal{T} \) if and only if \( \text{dGLR}_t(H) \) is bounded for all \( H \in \mathcal{H}(t) \).
Correspondence

Define the family of sets of probability measures \( \{ \hat{Q}^\gamma, \gamma > 0 \} \) as

\[
\hat{Q}^\gamma := \left\{ Q : \frac{dQ}{dP} = c(1 + \Lambda), \ c > 0, \ \Lambda \in \mathcal{L}^\gamma, \ c \mathbb{E}_P[1 + \Lambda] = 1 \right\},
\]

where

\[
\mathcal{L}^\gamma := \{ \Lambda : \Lambda \text{ is an } \mathcal{F}_T\text{-measurable r.v., } 0 \leq \Lambda \leq \gamma \}
\]

- \( \hat{Q} \) does not depend on time
- \( \hat{Q} \) is an increasing family of dynamically consistent sets of probability measures that corresponds/generates dGLR
- It satisfies all necessary technical assumptions for NGD theory above
Market Model Set-Up

Table: Bid price paths of underlying security $P^{bid}$

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
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</table>

- Ask price of underlying security: $P^{ask} := (1 + \lambda)P^{bid}$, where $\lambda$ is the transaction cost coefficient.
- Consider Arithmetic Asian European style call option with strike 75:

$\left(\frac{1}{3}\left(P_{mid}^0 + P_{mid}^1 + P_{mid}^2\right) - 75\right)^+$, where $P^{mid} := (P^{ask} + P^{bid})/2$. 
Table: Prices of an Arithmetic Asian Call Option with $\lambda = 0$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$S_0^{ask}$</th>
<th>$\Pi_0^{ask,\gamma}$</th>
<th>$\Pi_0^{bid,\gamma}$</th>
<th>$S_0^{bid}$</th>
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</table>
Asian Call Option

Liquidity Surface for Arithmetic Asian Call Option with Strike of 65

Bid-Ask Spread

TC Coefficient

Acceptability Level
Thank You!

The end of the talk ... but not of the story.