Asymmetric information, risk sharing and market liquidity

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Following Kyle (1985) we consider a market in continuous time in which one risky asset is exchanged for a riskless asset among three kinds of traders:

- **a single risk-neutral insider** with a private observation of the fundamental value, $V$, of the risky asset with the objective to maximise her expected profit.

- **uninformed noise traders** trade randomly for reasons exogenous to the model. The cumulative demand, $Z$, of the noise traders is given exogenously: $Z_t = \sigma B_t$, where $B$ is a standard Brownian motion.

- **market makers** who quote prices to provide supply against net demand of insider and noise traders. The prices are set conditional on their information and taking their risk aversion into account.
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For a survey of competition among market makers in market microstructure theory, see Biais, Glosten and Spatt (2005) in J. of FM.
Competition among the market makers

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The market makers have identical CARA utilities with the common risk aversion parameter $\rho$, and compete in a Bertrand fashion for the net demand of the risky asset. The number of market makers is assumed to be $N \geq 2$. 

In case of several market makers quoting the same winning price, we adopt the convention that the total order is equally split among them.

As a result of this competition, in equilibrium each market maker quotes the price which achieves zero expected-utility gain, i.e. utility from wealth is a martingale.
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How are the prices set?

- The market makers quote prices at time-$t$ of the form $H(t, Y_t)$, where $H$ is a smooth function and $Y_t = Z_t + X_t$ with $X$ being the trading strategy of the insider. We show that there exists a Markovian equilibrium so this assumption is not a loss of generality.
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- $H$ is strictly increasing in $Y$, i.e. $H_Y > 0$.
- As market makers are identical, they will offer the same quotes in the equilibrium and get an equal share of the total demand.
- As a result of this competition, in equilibrium each market maker quotes the price which achieves zero expected-utility gain, i.e.

$$\mathbb{E}[U(G_t) - U(G_s)|\mathcal{F}_s^Y] = 0, \forall s \leq t,$$

where

$$G_t := -\frac{1}{N} \int_0^t Y_s \, dH(s, Y_s) - \mathbf{1}_{t=1} \frac{Y_1}{N}(V - H(Y_1, 1)).$$
Definition of Equilibrium

Definition 1

A pair \((H^*, X^*)\) is an equilibrium if \(H^*\) is an admissible pricing rule, \(X^* \in \mathcal{A}(H^*)\), and the following conditions are satisfied:

1. **Market efficiency condition:** given \(X^*\), \(H^*\) satisfies zero-utility gain condition.

2. **Insider optimality condition:** given \(H^*\), \(X^*\) solves the insider optimisation problem:

\[
\mathbb{E}^\nu[W_{1}^{X^*}] = \sup_{X \in \mathcal{A}(H^*)} \mathbb{E}^\nu[W_{1}^{X}],
\]

where \(W_{1}^{X} := \mathbb{E}^\nu[(V - H(1, Y_1))X_1 + \int_0^1 X_s \, dH(s, Y_s)]\) and \(\mathbb{E}^\nu\) corresponds to the expectation using the probability measure of the insider who is given the realisation \(V = \nu\).
Suppose the market makers chooses an $H$ such that

$$H_t + \frac{\sigma^2}{2} H_{yy} = 0.$$ 

Following Back the trading strategies that are not absolutely continuous are necessarily suboptimal.
Optimality conditions for the insider’s problem

- Suppose the market makers chooses an $H$ such that $H_t + \frac{\sigma^2}{2} H_{yy} = 0$.

- Following Back the trading strategies that are not absolutely continuous are necessarily suboptimal.

- Therefore, it is enough to consider $X \in \mathcal{A}$ of the form $dX_t = \alpha_t dt$ for some measurable $\alpha$. With an abuse of notation, the insider’s problem becomes:

$$
\sup_{\alpha \in \mathcal{A}} \mathbb{E}^\nu \left[ \int_0^1 \{ V - H(t, Y_t) \} \alpha_t dt \right],
$$

(1)
Consider the function

\[ J(t, y) := \int_{\xi(t)}^{y} \{ H(t, u) - V \} \, du + \frac{1}{2} \sigma^2 \int_{t}^{1} H_y(s, \xi(s)) \, ds \]

where \( \xi(t) \) is the unique solution of \( H(t, \xi(t)) = V \).

Direct calculations lead to \( J_t + \frac{\sigma^2}{2} J_{yy} = 0 \) so that

\[
\sup_{X \in \mathcal{A}(H)} \mathbb{E}^V \left[ \int_{0}^{1} (V - H(t, Y_t)) \, dX_t \right] = \mathbb{E}^V [J(0, 0)] - \inf_{X \in \mathcal{A}(H)} \mathbb{E}^V [J(1, Y_1)].
\]

Thus, the optimal strategy for the insider is to find an absolutely continuous \( X \) so that \( H(1, Z_1 + X_1) = V \).
Market makers’ problem

- Now suppose that

\[ H_t + \frac{1}{2} \sigma^2 H_{yy} = 0 \]  

(2)

holds and there exists an absolutely continuous insider strategy so that \( H(1, Z_1 + X_1) = V \). Under what condition is this \( H \) an equilibrium pricing rule?
Market makers’ problem

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holds and there exists an absolutely continuous insider strategy so that $H(1, Z_1 + X_1) = V$. Under what condition is this $H$ an equilibrium pricing rule?

The zero expected utility gain for an individual market maker corresponds to $U(G)$ being an $(\mathcal{F}^Y, \mathbb{P})$-martingale. Recall that

$$G_t = -\frac{1}{N} \int_0^t Y_s \, dH(s, Y_s) - 1_{t=1} \frac{Y_1}{N} (V - H(Y_1, 1)).$$
On the other hand, in an equilibrium where (2) is satisfied and \( X_t = \int_0^t \alpha_s \, ds \) such that \( H(1, Y_1) = V \),

\[
U(G_t) = -\sigma U'(G_t) \frac{Y_t}{N} H_y(t, Y_t) dB^Y_t \\
- \frac{Y_t}{N} H_y(t, Y_t) U'(G_t) \left( \hat{\alpha}_t + \frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) \right) dt,
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where \( B^Y \) is an \( \mathcal{F}^Y \)-Brownian motion and \( \hat{\alpha}_t = \mathbb{E}[\alpha_t | \mathcal{F}_t^M] \).
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Thus, the zero-utility gain condition requires

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\]

Note that \( \hat{\alpha} \) is controlled by the insider so the insider should choose a strategy to ensure that the total demand in its own filtration looks like

\[
dY_t = \sigma dB^Y_t - \frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) \, dt.
\]
Leaving the Markovian structure aside, the way the price is settled in this market require the terminal price, $S_1$, to satisfy

$$
\mathbb{E} \left[ \exp \left( \frac{\rho}{N} (\mathcal{V} - S_1) \right) \bigg| \mathcal{F}_{Y_1} \right] = 1.
$$

As $U(G)$ must be a martingale in the equilibrium, a simple application of Ito’s formula yields

$$
dS_t = -\frac{\rho}{2N} Y_t \zeta_t^2 \, dt + \zeta_t \, dB_t^Y.
$$

As $Y$ is not bounded, the BSDE corresponding to $(S, \zeta)$ is not simply a quadratic BSDE.

The Markovian equilibrium that we will establish in the next slides will, therefore, provide a solution to the above BSDE.
Suppose that there exists a function $H$ such that

$$H_t + \frac{1}{2} \sigma^2 H_{yy} = 0$$

and for any Brownian motion $\beta$ there exists a solution to

$$d\xi_t = \sigma d\beta_t - \frac{\sigma^2 \rho}{2N} \xi_t H_y(t, \xi_t) \, dt$$

with ‘smooth’ transition density, $p$, and $H(1, \xi_1) \sim f(N(0, 1))$ when $\xi_0 = 0$. 

**Equilibrium**

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Liquidity and risk aversion
Suppose that there exists a function $H$ such that

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$$d\xi_t = \sigma d\beta_t - \frac{\sigma^2 \rho}{2N} \xi_t H_y(t, \xi_t) \, dt$$ (4)

with ‘smooth’ transition density, $p$, and $H(1, \xi_1) \sim f(N(0, 1))$ when $\xi_0 = 0$.

Then, this $H$ and

$$\alpha_t = -\frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) + \frac{p_y}{p}(t, Y_t; 1, H^{-1}(1, V))$$

constitute an equilibrium.
Why is this a candidate for equilibrium?

Indeed, if

$$dY_t = \sigma dB_t + \left\{ -\frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) + \frac{p_y}{p}(t, Y_t; 1, H^{-1}(1, V)) \right\} dt, \quad (5)$$

then by standard results on filtering

$$dY_t = \sigma dB_t^Y - \frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) \, dt$$

$$+ \int_{\mathbb{R}} \frac{p_y}{p}(t, Y_t; 1, z) \mathbb{P}(H^{-1}(1, V) \in dz | \mathcal{F}_t^Y) \, dt.$$

Moreover, one can show that

$$\mathbb{P}(H^{-1}(1, V) \in dz | \mathcal{F}_t^Y) = p(t, Y_t; 1, z) \, dz$$

so that

$$dY_t = \sigma dB_t^Y - \frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) \, dt.$$
One way to see the above conditional distribution is to notice that, conditional on $H^{-1}(1, V) = z$, the distribution of $Y$ defined via (5) is that of $\xi$ from (4) conditioned to equal $z$ at time 1.

This also explains why $Y_1 = H^{-1}(1, V)$, i.e. $H(1, Y_1) = V$. 
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One can view what’s going on as a filtration enlargement.

Observe that in the market’s filtration

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An h-transform and a filtration expansion

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The insider’s filtration is the market filtration enlarged with the random variable \( V \). Since \( H(1, Y_1) = V \) in this equilibrium, the well-known formulas for initial enlargement of filtrations yield

\[
dY_t = \sigma dB_t + \left\{ -\frac{\sigma^2 \rho}{2N} Y_t H_y(t, Y_t) + \frac{\rho_y}{p}(t, Y_t; 1, H^{-1}(1, V)) \right\} \, dt.
\]
The conditions on $\xi$ and $H$ imply that the boundary condition for $H$ is given by

$$H(1, y) = f(\Phi^{-1}(P_1(y)))$$

where $\Phi$ is the cdf of a standard normal, $P_1$ is the cdf of $\xi_1$, and $\xi$ solves (4) with $\xi_0 = 0$. 
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If $f$ is bounded with a continuous derivative, then it can be shown by a combination of a fixed point argument and Girsanov transform that such a pair $(H, \xi)$ exists. However, it does not seem possible to find $H$ explicitly in general.
Start from a probability distribution on \( \mathbb{R} \) with a density belonging to a suitably chosen closed convex subset, \( D \), of bounded continuous functions endowed with the sup-norm.

Consider the operator \( T \) which maps \( P \in D \) to the distribution of \( \xi_1 \), which solves (4), where \( H \) is determined via \( H(1, y) = f(\Phi^{-1}(P(y))) \). The way we fix \( D \) leads to \( TD \subset D \) and \( TD \) is an equicontinuous family of functions.

Show that \( T \) is a continuous operator and \( TD \) is pre-compact using Arzela-Ascoli Theorem and the fact that the tails of \( P \in D \) can be made arbitrarily small uniformly.

Apply Schauder’s Fixed Point Theorem.
In case \( f(y) = ay + b \), we can explicitly compute the equilibrium.

The naive approach for a solution is to start with \( H(t, y) = \lambda y + b \) for some constant \( \lambda \) to be determined so that \( \xi \) becomes a Gaussian process:

\[
\xi_t = \sigma \beta_t - \sigma^2 \frac{\lambda \rho}{2N} \int_0^t \xi_s \, ds.
\]

Then, \( \xi_1 \) is normal with mean 0 and variance

\[
\frac{N}{\rho \lambda} \left( 1 - \exp \left( -\frac{\rho \sigma^2 \lambda}{N} \right) \right).
\]
Since $H(1, \xi_1) \overset{d}{=} f(\eta)$, we obtain $\lambda$ as the unique solution to

$$1 - e^{-\frac{\rho \sigma^2}{N} \lambda} = \frac{\rho a^2}{N} \frac{1}{\lambda},$$

and the equilibrium demand $Y^*$ solves

$$dY_t^* = \sigma dB_t + \frac{\rho \sigma^2}{2N} a \eta - \lambda Y_t^* \cosh \left( \frac{\rho \sigma^2 \lambda}{2N} (1 - t) \right) \frac{1}{\sinh \left( \frac{\rho \sigma^2 \lambda}{2N} (1 - t) \right)}.$$  (7)

It is immediate from (6) that $\lambda > a^2 \rho^0$ where $\rho^0 := \rho/N$. Moreover, it can be shown that $\lambda$ is increasing in $\rho^0$. 
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Consequently, as the market makers become more risk averse, the market becomes less liquid.
Total order flow has a drift in its own filtration:

\[
dY_t^* = \sigma dB_t^Y - \frac{\sigma^2 \rho}{2N} Y_t^* H_y^*(t, Y_t^*) \, dt,
\]

i.e. insider’s trades are no longer inconspicuous.
First implications of equilibrium

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- The equilibrium total order process is mean reverting, i.e. the large buy orders are followed by sell orders, and vice versa. This is a result of risk sharing between the market makers and the insider.
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As \( \frac{\rho}{N} \to 0 \) the linear equilibrium in (7) converges to

\[ dY_t^* = \sigma dB_t + \sigma \frac{\eta - \frac{Y_t^*}{\sigma}}{1 - t} \, dt \]

and \( \lambda \to \frac{a}{\sigma} \).
To obtain the result of these models we set \( \rho = 0, \ a = 1 \) in the linear case.
Benchmark model I: Kyle/Back model

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- The profit of the insider is $\sigma$.

- The resilience is the speed with which the prices converge to the fundamental value. In Kyle’s equilibrium it equals

$$\frac{1}{1 - t} \left( - \frac{d}{dt} \left( \frac{V - E^Y[\lambda Y_t^*]}{V - E^Y[\lambda Y_t^*]} \right) \right).$$
Still in the linear case but the strategic trader has no private information.
Benchmark model II: strategic trader model

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- In turns out that in the equilibrium, the total order flow, \( Y^* \), solves

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dY^*_t = dB_t - \frac{\sigma^2 \rho^2}{8N^2} Y^*_t \, dt - 1_{t=1} \frac{Y^*_1 \cdot 1}{2}.
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- The equilibrium pricing rule is

$$H(t, y) := \begin{cases} 
  b + \frac{\rho}{2N} y, & t = 1; \\
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- The profit of the strategic trader is given by $\frac{\rho \sigma^2}{8N}$. 

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Liquidity and risk aversion
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\end{cases}.$$

- The profit of the strategic trader is given by $\frac{\rho \sigma^2}{8N}$.
- The market depth is $\lambda^0 := \frac{4N}{\rho}$. 
The depth of the market is still constant and equal to $\frac{1}{\lambda}$ where $\lambda$ solves (6).

Market becomes less liquid as the market makers become more risk averse. The panel on the right depicts how this loss of liquidity, measured in terms of the relative depth, i.e. $\frac{1}{\lambda \sigma}$, depends on the risk aversion and noise volatility.
Resilience is given by

$$
\frac{d}{dt} \log (\eta - \mathbb{E}^\nu [\lambda Y_t^*]) = \frac{\lambda \mu \cosh(\lambda t)}{\sinh(\lambda \mu) - \sinh(\lambda \mu t)}, \quad \mu : = \frac{\rho \sigma^2}{2N}.
$$

For a better comparison the resilience parameter is normalised by its counterpart when the market makers are risk neutral; \( \sigma = 0.5 \).
Efficiency is a measurement of how informative the market prices are. Using the explicit form of the Ornstein-Uhlenbeck processes, we get

\[ \Sigma(t) = \text{Var}(\eta|\mathcal{F}_t^M) = \frac{1 - \exp \left( -\frac{\lambda \sigma^2 \rho}{N} (1 - t) \right)}{1 - \exp \left( -\frac{\lambda \sigma^2 \rho}{N} \right)} \geq 1 - t. \]

The loss of efficiency is monotone in \( \rho \) and \( \sigma \). This is in contrast with the risk neutral case where the efficiency is independent of the volatility of the noise trading.
The straight line below the other curves is the efficiency when the market makers are risk neutral. In the left pane $\sigma$ is taken to be 0.5 while $\rho^0 = 1$ on the right.
When the market makers become risk averse, the prices exhibit a reversal. In particular,

\[ M(s) := \lim_{t-s=\varepsilon, u-s=\varepsilon, \varepsilon \to 0} \frac{\text{Cov}(H(t, Y_t) - H(s, Y_s), H(u, Y_u) - H(t, Y_t))}{\sqrt{\text{Var}(H(t, Y_t) - H(s, Y_s)) \text{Var}(H(u, Y_u) - H(t, Y_t))}} \varepsilon \]

\[ = -\frac{\lambda \mu}{2N} \left(1 + \exp \left(-\frac{\lambda \sigma^2 \rho}{N} s\right)\right). \]
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\]

\[
= -\frac{\lambda \mu}{2N} \left(1 + \exp \left(-\frac{\lambda \sigma^2 \rho}{N} s\right)\right).
\]

Similar calculations yield that when a strategic trader has no private information

\[
M(s) = -\frac{\lambda^0 \mu}{2} \left(1 + \exp \left(-\lambda^0 \sigma^2 \rho^0 s\right)\right),
\]

where \(\rho^0 := \frac{\rho}{N}, \lambda^0 = \frac{\rho}{4N},\) and \(\mu = \frac{\sigma^2 \rho}{2N}.\)
Both plots above assume $\sigma = 1$, and show monotonic behaviour of price reversal as a function of time and risk aversion. The right pane plots price reversal as a function of time and risk aversion for the case of strategic trader equilibrium, whereas the left one plots it for the case of the insider equilibrium.
The ex-ante profit of the insider is found to be $\frac{1+\sigma^2\lambda^2}{2\lambda}$. Thus, the change in the ex-ante profits due to the risk aversion is equal to $\frac{(1-\lambda\sigma)^2}{2\lambda} > 0$. 
The ex-ante profit of the insider is found to be $1 + \sigma^2 \lambda^2$. Thus, the change in the ex-ante profits due to the risk aversion is equal to $\left(1 - \frac{\lambda \sigma}{2\lambda}\right)^2 > 0$.

The excess profits for the insider is monotonically increasing to infinity in both risk aversion and noise volatility. This implies that the noise traders lose more in the equilibrium with higher risk aversion of market makers.
Insider’s profits increase as the market makers get more risk averse. Excess normalised profits, measured by $\frac{(1-\sigma \lambda)^2}{2\lambda \sigma}$, increase, too.
The difference between the ex-ante profit of the insider and that of the strategic trader is the value of information and it equals to

\[
\frac{1}{2\lambda} + \frac{\sigma^2}{2}(\lambda - \lambda^0).
\]

\[
\sigma = 0.5, \quad \sigma = 0.75, \quad \sigma = 1.
\]
Conclusion

Risk aversion causes

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Conclusion

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- smaller depth,
- prices converge to the fundamental value at a smaller rate,
- less efficient markets,
- noise traders lose more money on average,
- insider poses herself as a strategic trader with no private information in the equilibrium,
- mean reverting total order flow and price reversal.