Optimization Problems with Stochastic Order Constraints

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Motivation

Risk-Averse Optimization Models

Choose a decision $z \in Z$, which results in a random outcome $G(z) \in \mathcal{L}_p(\Omega, \mathcal{F}, P)$ with “good” characteristics paying special attention to low-probability-high-impact events.

- **Utility models** apply a nonlinear transformation to the realizations of $G(z)$ (expected utility) or to the probability of events (rank dependent utility/distortion). Expected utility models optimize $\mathbb{E}[u(G(z))]$.

- **Probabilistic / chance constraints** impose prescribed probability on some events: $\mathbb{P}[G(z) \geq \eta]$.

- **Mean–risk models** optimize a composite objective of the expected performance and a scalar measure of undesirable realizations $\mathbb{E}[G(z)] - \varrho[G(z)]$ (risk/ deviation measures).

- **Stochastic ordering constraints** compare random outcomes using stochastic orders and random benchmarks.
1 Stochastic orders

2 Stochastic orders as constraints

3 Optimality conditions and duality
   • Relation to von Neumann utility theory
   • Relation to rank dependent utility
   • Relation to coherent measures of risk

4 Multivariate and Dynamic Orders

5 Numerical methods

6 Applications
   • Portfolio optimization
   • Beyond portfolio optimization
For $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$

$$X \succeq_{\mathcal{F}} Y \iff \int_{\Omega} u(X(\omega)) P(d\omega) \geq \int_{\Omega} u(Y(\omega)) P(d\omega) \quad \forall u(\cdot) \in \mathcal{F}$$

Collection of functions $\mathcal{F}$ is the generator of the order.

**Generators**

- $\mathcal{F}_1 = \{\text{nondecreasing functions } u : \mathbb{R} \to \mathbb{R}\}$ generates the usual stochastic order or first order stochastic dominance ($X \succeq_{(1)} Y$)
  Mann and Whitney (1947), Blackwell (1953), Lehmann (1955)

- $\mathcal{F}_2 = \{\text{nondecreasing concave } u : \mathbb{R} \to \mathbb{R}\}$ generates the second order stochastic dominance relation ($X \succeq_{(2)} Y$)
  Quirk and Saposnik (1962), Fishburn (1964), Hadar and Russell (1969)

- $\tilde{\mathcal{F}}_2 = \{\text{nondecreasing convex } u : \mathbb{R} \to \mathbb{R}\}$ generates the increasing convex order ($X \preceq_{\text{ic}} Y$) counterpart of stochastic dominance of second order when small values are preferred
For any $X \in L_k(\Omega, \mathcal{F}, P)(\Omega, \mathcal{F}, \mathbb{P})$, we define

**Distribution Functions**

\[
F_1(X; \eta) = \int_{-\infty}^{\eta} P_X(dt) = \mathbb{P}\{X \leq \eta\} \quad \text{for all } \eta \in \mathbb{R}
\]

\[
F_k(X; \eta) = \int_{-\infty}^{\eta} F_{k-1}(X; t) dt \quad \text{for all } \eta \in \mathbb{R}, \quad k = 2, 3, \ldots
\]

The function $F^{(k)}_X$ is nondecreasing for $k \geq 1$ and convex for $k \geq 2$.

**Quantile function**

\[
F_{-1}(X; p) = \inf\{\eta : F_1(X; \eta) \geq p\}, \quad p \in (0, 1)
\]

**Survival function**

\[
\overline{F}_1(X; \eta) = 1 - F_1(X; \eta) = \mathbb{P}\{X > \eta\}, \quad \eta \in \mathbb{R}
\]
The usual stochastic order

\[ X \succeq_{(1)} Y \iff F_1(X; \eta) \leq F_1(Y; \eta) \quad \text{for all } \eta \in \mathbb{R} \]

\[ \iff F_{(-1)}(X; p) \geq F_{(-1)}(Y; p) \quad \text{for all } 0 < p < 1. \]

\[ \iff \overline{F}_1(X; \eta) \geq \overline{F}_1(Y; \eta) \quad \text{for all } \eta \in \mathbb{R}. \]
Second-Order Stochastic Dominance

\[ F_2(X; \eta) = \int_{-\infty}^{\eta} F_1(X; t) \, dt = \mathbb{E}\left[ \max(0, \eta - X) \right] \text{ for all } \eta \in \mathbb{R} \]

\[ X \succeq_{(2)} Y \iff F_2(X; \eta) \leq F_2(Y; \eta) \text{ for all } \eta \in \mathbb{R} \]

\[ \iff \mathbb{E}\left[ \max(0, \eta - X) \right] \leq \mathbb{E}\left[ \max(0, \eta - Y) \right] \text{ for all } \eta \in \mathbb{R} \]
Higher order relation

For any \( X \in \mathcal{L}_k(\Omega, \mathcal{F}, P)(\Omega, \mathcal{F}, \mathbb{P}) \), \( \|X\|_k = \left( \mathbb{E}(|X|^k) \right)^{\frac{1}{k}} \) and

\[
F_{(k+1)}(X, \eta) = \frac{1}{k!} \int_{-\infty}^{\eta} (\eta - t)^k P_X(dt) = \frac{1}{k!} \| \max(0, \eta - X) \|_k^k \quad \forall \eta \in \mathbb{R},
\]

**kth degree Stochastic Dominance (kSD), \( k \geq 2 \)**

\[
X \succeq_{(k)} Y \iff F_k(X, \eta) \leq F_k(Y, \eta) \quad \text{for all} \quad \eta \in \mathbb{R},
\]

\[
\| \max(0, \eta - X) \|_{k-1}^k \leq \| \max(0, \eta - Y) \|_{k-1}^{k-1}
\]

**The generator**

\( \mathcal{F}_k \) contains all functions \( u : \mathbb{R} \to \mathbb{R} \) such that a non-increasing, left-continuous, bounded function \( \varphi : \mathbb{R} \to \mathbb{R}_+ \) exists such that \( u^{(k-1)}(\eta) = (-1)^k \varphi(\eta) \) for a.a. \( \eta \in \mathbb{R} \).
Second Order Dominance and Inverse Distribution Functions

Absolute Lorenz function (Max Otto Lorenz, 1905)

\[
F_{(-2)}(X; p) = \int_0^p F_{(-1)}(X; t) \, dt \quad \text{for } 0 < p \leq 1,
\]

\[F_{(-2)}(X; 0) = 0 \quad \text{and} \quad F_{(-2)}(X; p) = +\infty \quad \text{for } p \notin [0, 1].\]

Fenchel conjugate function of \(F\):

\[F^*(p) = \sup_u \{pu - F(u)\}\]

Lorenz function and Expected shortfall are Fenchel conjugates

\[F_{(-2)}(X; \cdot) = [F_2(X; \cdot)]^* \quad \text{and} \quad F_2(X; \cdot) = [F_{(-2)}(X; \cdot)]^*\]

Ogryczak - Ruszczyński (2002)

Second order dominance \(\equiv\) Relation between Lorenz function

\[X \succeq_{(2)} Y \iff F_{(-2)}(X; p) \geq F_{(-2)}(Y; p) \quad \text{for all } 0 \leq p \leq 1.\]
Characterization of Stochastic Dominance by Lorenz Functions

\[ X \succeq_2 Y \iff F_{(-2)}(X; p) \geq F_{(-2)}(Y; p) \quad \text{for all} \quad 0 \leq p \leq 1. \]
Preference to small values: Increasing convex order

Characterization via integrated survival function

For $X, Y \in \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $X$ is smaller than $Y$ ($X \preceq_{ic} Y$) if and only if

$$
\int_{\eta}^{\infty} P(X > t) \, dt \leq \int_{\eta}^{\infty} P(Y > t) \, dt \quad \text{for all } \eta \in \mathbb{R}.
$$

The excess function and its Fenchel conjugate

$$
H(Z, \eta) = \int_{\eta}^{\infty} F(Z, t) \, dt = \mathbb{E}(Z - \eta) +
$$

$$
L(Z, q) = -\int_{1+q}^{1} F(-1)(Z, t) \, dt \quad \text{for } -1 \leq q < 0,
$$

$L(Z, 0) = 0, L(Z, q) = \infty$ for $q \notin [-1, 0]$

Increasing convex order vs. Second order dominance

$$
X \preceq_{ic} Y \iff -X \succeq_{(2)} -Y.
$$
\( \mathcal{W}_1 \) contains all continuous nondecreasing functions \( w : [0, 1] \to \mathbb{R} \).
\( \mathcal{W}_2 \subset \mathcal{W}_1 \) contains all concave subdifferentiable at 0 functions.

**Theorem [DD, A. Ruszczyński, 2006]**

(i) For all random variables \( X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P) \) the relation \( X \preceq_{(1)} Y \) holds if and only if for all \( w \in \mathcal{W}_1 \)

\[
\int_0^1 F_{(-1)}(X; p) \, dw(p) \geq \int_0^1 F_{(-1)}(Y; p) \, dw(p) \quad (1)
\]

(ii) \( X \succeq_{(2)} Y \) holds if and only if (1) is satisfied for all \( w \in \mathcal{W}_2 \).

**Corollary**

\( X \preceq_{ic} Y \) holds if and only if (1) is satisfied for all convex functions \( w \) which are subdifferentiable at zero.

Quiggin (1982), Schmeidler (1986–89), Yaari (1987)
Acceptance Sets

For all \( k \geq 1 \), \( Y \) - benchmark outcome in \( \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) \), \( [a, b] \subseteq \mathbb{R} \).

Acceptance sets \( A_k(Y; [a, b]) = \{ X \in \mathcal{L}_{k-1} : X \succeq_{(k)} Y \text{ in } [a, b] \} \)

**Theorem**

The set \( A_k(Y; [a, b]) \) is **convex and closed** for all \( [a, b] \), all \( Y \), and \( k \geq 2 \). Its recession cone has the form

\[
A_k^\infty(Y; [a, b]) = \{ H \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) : H \geq 0 \text{ a.s. on } [a, b] \}
\]

\( A_1(Y; [a, b]) \) is **closed** and \( A_k(Y; [a, b]) \subseteq A_{k+1}(Y; [a, b]) \quad \forall k \geq 1 \). \( A_k(Y; [a, b]) \) is a cone pointed at \( Y \) if and only if \( Y \) is a constant in \( [a, b] \).

**Theorem**

If \((\Omega, \mathcal{F}, P)\) is atomless, then \( A_2(Y; \mathbb{R}) = \overline{\text{co}} A_1(Y; \mathbb{R}) \) if \( \Omega = \{1..N\} \), and \( P[k] = 1/N \), then \( A_2(Y; \mathbb{R}) = \text{co} A_1(Y; \mathbb{R}) \)

*The result is not true for general probability spaces*
Dominance Relation in Optimization

Introduced by Dentcheva and Ruszczyński in 2003

\[
\min f(z)
\]
\[\begin{align*}
(P) & \quad \text{s.t. } G_i(z) \succeq_{(k_i)} Y_i & i = 1, \ldots, m, \\
& \quad z \in Z.
\end{align*}\]

\(Y_i\) - benchmark random outcome

\(Z\) - convex subset of a separable Banach space \(\mathcal{H}\),
\(G_i\) – continuous operators from \(\mathcal{H}\) to the space \(\mathcal{L}_{k_i-1}(\Omega, \mathcal{F}, P; \mathbb{R})\),
\(k_i \geq 1\), \(f\) – continuous function defined on \(\mathcal{H}\).

The stochastic order constraints reflect risk aversion

- \(G_i(z)\) is preferred over \(Y_i\) by all risk-averse decision makers with utility functions in the generator \(\mathcal{F}_{k_i}\);
- Easier consensus on a benchmark rather than a utility function;
- Data of a benchmark is readily available.
Assets $j = 1, \ldots, n$ with random return rates $R_j$
Reference return rate $Y$ (e.g. index, existing portfolio, etc.)
Decision variables $z_j$, $j = 1, \ldots, n$, $Z$ -polyhedral set
Portfolio return rate $R(z) = \sum_{j=1}^{n} z_j R_j$

$$\max \ f(z)$$

s.t. $\sum_{j=1}^{n} z_j R_j \succeq Y$

$z \in Z$

$$f(x) = \mathbb{E}[R(x)] \text{ or } f(x) = -\varrho[R(x)]: \text{ measure of risk.}$$
All Statements are Equivalent

\[ \sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y \]

\[ F_{(-2)} \left( \sum_{j=1}^{n} z_j R_j ; p \right) \geq F_{(-2)} (Y ; p) \text{ for all } p \in [0, 1] \]

continuum of CVaR constraints for every risk level \( p \in [0, 1] \)

\[ \mathbb{E} u \left( \sum_{j=1}^{n} z_j R_j \right) \geq \mathbb{E} u (Y) \]

for all concave nondecreasing functions \( u \) (von Neuman-Morgenstern utility)

\[ \int_{0}^{1} F_{(-1)} \left( \sum_{j=1}^{n} z_j R_j ; p \right) d w(p) \geq \int_{0}^{1} F_{(-1)} (Y ; p) d w(p) \]

for all concave nondecreasing functions \( w \) (rank dependent utility)
Second Order Dominance Constraints

Given $Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ - benchmark random outcome

**Primal Stochastic Dominance Constraints**

$$\max f(z)$$  
subject to $F_2(G(z); \eta) \leq F_2(Y; \eta)$, $\forall \eta \in [a, b]$, $z \in Z$

**Inverse Stochastic Dominance Constraints**

$$\max f(z)$$  
subject to $F_2(G(z); p) \geq F_2(Y; p)$, $\forall p \in [\alpha, \beta]$, $z \in Z$

$Z$ is a closed subset of a Banach space $\mathcal{X}$, $[\alpha, \beta] \subset (0, 1)$, $[a, b] \subset \mathbb{R}$

$G : \mathcal{X} \rightarrow \mathcal{L}_1(\Omega, \mathcal{F}, P)$ is continuous and for $P$-almost all $\omega \in \Omega$ the functions $[G(\cdot)](\omega)$ are concave and continuous

$f : \mathcal{X} \rightarrow \mathbb{R}$ is concave and continuous
The Lagrangian-like functional $L : \mathcal{X} \times \mathcal{F}_2([a, b]) \to \mathbb{R}$

$$L(z, u) := f(z) + \mathbb{E}[u(G(z)) - u(Y)]$$

$\mathcal{F}_2([a, b])$ modified generator.

**Uniform Dominance Condition (UDC) for problem ($\mathcal{P}_2$)**
A point $\tilde{z} \in Z$ exists such that
$$\inf_{\eta \in [a, b]} \{ F_2(Y; \eta) - F_2(G(\tilde{z}); \eta) \} > 0.$$

**Theorem** Assume UDC. If $\hat{z}$ is an optimal solution of ($\mathcal{P}_2$) then $\hat{u} \in \mathcal{F}_2([a, b])$ exists:

1. $$L(\hat{z}, \hat{u}) = \max_{z \in Z} L(z, \hat{u})$$ (2)
2. $$\mathbb{E}[\hat{u}(G(\hat{z}))] = \mathbb{E}[\hat{u}(Y)]$$ (3)

If for some $\hat{u} \in \mathcal{F}_2([a, b])$ an optimal solution $\hat{z}$ of (2) satisfies the dominance constraints and (3), then $\hat{z}$ solves ($\mathcal{P}_2$).
Lagrangian-like functional $\Phi : \mathcal{Z} \times \mathcal{W}([\alpha, \beta]) \rightarrow \mathbb{R}$

$$\Phi(z, w) = f(z) + \int_{0}^{1} F_{(-1)}(G(z); p) \, dw(p) - \int_{0}^{1} F_{(-1)}(Y; p) \, dw(p)$$

$\mathcal{W}([\alpha, \beta])$ is the modified generator of the relaxed order

Uniform inverse dominance condition (UIDC) for (P$_{-2}$)

$\exists \tilde{z} \in Z$ such that $\inf_{p \in [\alpha, \beta]} \left\{ F_{(-2)}(G(\tilde{z}); p) - F_{(-2)}(Y; p) \right\} > 0.$

**Theorem**

Assume UIDC. If $\hat{z}$ is a solution of (P$_{-2}$), then $\hat{w} \in \mathcal{W}([\alpha, \beta])$ exists:

$$\Phi(\hat{z}, \hat{w}) = \max_{z \in Z} \Phi(z, \hat{w})$$ (4)

$$\int_{0}^{1} F_{(-1)}(G(\hat{z}); p) \, d\hat{w}(p) = \int_{0}^{1} F_{(-1)}(Y; p) \, d\hat{w}(p)$$ (5)

If for some $\hat{w} \in \mathcal{W}([\alpha, \beta])$ and a solution $\hat{z}$ of (4) the dominance constraint and (5) are satisfied, then $\hat{z}$ is a solution of (P$_{-2}$).
Duality Relations to Utility Theories

The Dual Functionals

\[ D(u) = \sup_{z \in Z} L(z, u) \quad \psi(w) = \sup_{z \in Z} \Phi(z, w) \]

The Dual Problems

\[ (\mathcal{D}_2) \quad \min_{u \in \mathcal{U}_2([a, b])} D(u) \quad (\mathcal{D}'_2) \quad \min_{w \in \mathcal{W}([\alpha, \beta])} \psi(w). \]

Theorem

Under UDC/UIDC, if problem \((\mathcal{P}_2)\) resp. \((\mathcal{P}'_2)\) has an optimal solution, then the corresponding dual problem has an optimal solution and the same optimal value. The optimal solutions of the dual problem \((\mathcal{D}_2)\) are the utility functions \(\hat{u} \in \mathcal{U}_2([a, b])\) satisfying (2)–(3) for an optimal solution \(\hat{z}\) of problem \((\mathcal{P}_2)\). The optimal solutions of \((\mathcal{D}'_2)\) are the rank dependent utility functions \(\hat{w} \in \mathcal{W}([\alpha, \beta])\) satisfying (4)–(5) for an optimal solution \(\hat{z}\) of problem \((\mathcal{P}'_2)\).
A coherent measure of risk is a functional $\varrho : \mathcal{L}_1(\Omega, \mathcal{F}, P) \rightarrow \overline{\mathbb{R}}$ satisfying the axioms:

- **Convexity:** $\varrho(\alpha X + (1 - \alpha) Y) \leq \alpha \varrho(X) + (1 - \alpha) \varrho(Y)$ for all $X, Y \in \mathcal{L}_1$, $\forall \alpha \in [0, 1]$.
- **Monotonicity:** If $Y(\omega) \geq X(\omega)$ $\forall \omega \in \Omega$, then $\varrho(Y) \leq \varrho(X)$.
- **Translation Equivariance:** $\varrho(X + a) = \varrho(X) - a$ $\forall a \in \mathbb{R}$.
- **Positive homogeneity:** $\varrho(tX) = t\varrho(X)$ $\forall t > 0$.

$$\max_{z, \sigma} \{f(z) - \lambda \sigma : z \in Z, \ G(z) + \sigma \succeq Y\}$$

$\lambda > 0$ is a tradeoff between $f(\cdot)$ and the error in dominating.

**Proposition**

The optimal value of $\sigma$ is a coherent measure of risk.
Mean-risk models as Lagrangian Relaxation

Kusuoka representation

If $\Omega$ is atomless, then for every law invariant, finite-valued coherent measure of risk on $L_\infty(\Omega, \mathcal{F}, P)$ a convex set $\mathcal{M}_\varrho$ of probability measures on $(0, 1]$ exists such that

$$\varrho(X) = \sup_{\mu \in \mathcal{M}_\varrho} \left( - \int_0^1 \frac{1}{p} F(-2)(X; p) \mu(dp) \right) \quad \forall X \in L_\infty.$$ 

Theorem

Under the UIDC, if $\hat{z}$ is an optimal solution of $(\mathcal{P}_{-2})$, then a law-invariant coherent risk measure $\hat{\varrho}$ and $\kappa \geq 0$ exist such that $G(\hat{z})$ is a solution of the mean-risk problem

$$\max_{z \in Z} \{ f(z) - \kappa \hat{\varrho}(G(z)) \} \quad \text{and} \quad \kappa \hat{\varrho}(G(\hat{z})) = \kappa \hat{\varrho}(Y).$$

Moreover, $\mathcal{M}_\varrho$ is singleton.

If the dominance constraint is active, then $\hat{\varrho}(G(\hat{z})) = \hat{\varrho}(Y)$. 
The Implied Dominance Constraint

Given the problem

\[ \text{(R)} \quad \max_{X \in C} \{ f(X) - \kappa \varrho(X) \} \]

\( \varrho(\cdot) \) a coherent law invariant measure of risk and \( \kappa > 0 \)

\( \text{rca}([0, 1]) \) - space of regular countably additive measures on \([0, 1]\).

**Theorem** If \( M_\varrho \) is compact* in \( \text{rca}([0, 1]) \) and \( \hat{X} \) is a solution of problem \( \text{(R)} \), then \( \exists \hat{\mu} \in M \) such that

\[ \varrho(\hat{X}) = -\int_0^1 \frac{1}{p} F_{(-2)}(\hat{X}; p) \hat{\mu}(dt), \]

and for every \( Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P) \) satisfying the conditions

\[ F_{(-2)}(Y; t) \leq F_{(-2)}(\hat{X}; t), \quad \text{for all} \quad t \in [0, 1], \]

\[ F_{(-2)}(Y; t) = F_{(-2)}(\hat{X}; t), \quad \text{for all} \quad t \in \supp(\hat{\mu}), \]

the point \( \hat{X} \) is also a solution of problem \( \text{(P}_{-2}) \) with \([\alpha, \beta] = [0, 1]\).
Assets $j = 1, \ldots, n$, $n = 719$ with random returns $R_j$
Decision variables $z_j$, $j = 1, \ldots, n$, $Z$-simplex
Portfolio return $G(z) = \sum_{j=1}^{N} z_j R_j$
Reference return $Y$ is the Standard and Poor 500 index.

$$\max \mathbb{E}[\sum_{j=1}^{n} z_j R_j]$$

subject to $\sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y$
$z \in Z$

We use 248 realizations of the joint returns

The optimal portfolio

7 stocks with weights 10.98%, 7.08%, 21.79%, 13.19%, 36.51%, 4.41% 6.04%, correspondingly; the expected return is 0.64% vs. -0.0359% of S&P 500
Implied Expected Utility

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Optimization and Stochastic Orders
Implied Rank Dependent Utility Function

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\[ \varrho(X) = 0.1069 \text{AVaR}_{0.1772}(X) + 0.014 \text{AVaR}_{0.3102}(X) + 0.0274 \text{AVaR}_{0.3636}(X) \\
+ 0.0577 \text{AVaR}_{0.4093}(X) + 0.3073 \text{AVaR}_{0.4594}(X) + 0.2935 \text{AVaR}_{0.4967}(X) \\
+ 0.1077 \text{AVaR}_{0.5081}(X) + 0.0576 \text{AVaR}_{0.557}(X) + 0.0213 \text{AVaR}_{0.5647}(X) \\
+ 0.0066 \text{AVaR}_{0.575}(X) \]
Consider $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ in $\mathcal{L}_1^m(\Omega, \mathcal{F}, P)$. 

### Coordinate Order

$X \succeq^{\text{sep}}_{(2)} Y \iff X_t \succeq^{(2)} Y_t, \ t = 1, \ldots, m$

Generator $\mathcal{F}_s$ all functions $u(X) = \sum_{i=1}^m u_i(X_i)$ with concave nondecreasing $u_i : \mathbb{R} \rightarrow \mathbb{R}$. Our earlier analysis covers this case. Ignores temporal structure and dependency.

### Increasing Convex Order

$X \succeq_{(\text{icx})} Y \iff \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)] \ \forall \ u \in \mathcal{F}$

Generator $\mathcal{F}_a -$ all concave nondecreasing functions $u : \mathbb{R}^m \rightarrow \mathbb{R}$

Hard to treat analytically, the generator is too rich.

### Proposed approach

Define the multivariate order via a family of univariate orders.
A random vector $X \in \mathcal{L}_1^m$ dominates a random vector $Y \in \mathcal{L}_1^m$ with respect to the linear second-order dominance $(X \succeq_{\text{lin}}^{(2)} Y)$ if $c^\top X \succeq_{\text{lin}}^{(2)} c^\top Y$ for all $c \in S$, where $S = \{ c \in \mathbb{R}_+^m : \|c\|_1 = 1 \}$.

If the set $S$ contains non-increasing sequences, then $\succeq_{\text{lin}}^{(2)}$ can be used to compare sequences.

The linear order $\succeq_{\text{lin}}^{(2)}$ implies the coordinate order $X_i \succeq_{(2)} Y_i$, $i = 1, \ldots, m$ but is not equivalent to it.

Other definitions: A. Müller, D. Stoyan, Homem-de-Mello and Mehrotra: $S$ is a polyhedron, or a compact convex set.
The set $\mathcal{D}$ contains all mappings $Q : S \to \mathcal{F}$ such that $(c, x) \to [Q(c)](c^T x)$ is Lebesgue measurable on $S \times \mathbb{R}^m$.

$\mathcal{M}(S)$ is the space of regular countably additive measures on $S$ with finite variation;

$\mathcal{M}_+(S)$ is its subset of nonnegative measures.

With every mapping $Q \in \mathcal{D}$ and every finite measure $\mu \in \mathcal{M}_+(S)$ we associate a function $\varphi_{Q, \mu} : \mathbb{R}^m \to \mathbb{R}$ as follows:

$$\varphi_{Q, \mu}(x) = \int_S [Q(c)](c^T x) \mu(dc).$$

We define the class of functions $\mathcal{F}_m = \{\varphi_{Q, \mu} : Q \in \mathcal{D}, \mu \in \mathcal{M}_+\}$.

**Theorem**

For each $X, Y \in \mathcal{L}_1^m$ the relation $X \succeq_{\text{lin}}^{(2)} Y$ is equivalent to

$$\mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)] \quad \text{for all } \varphi \in \mathcal{F}_m.$$
Control problem with order constraint and its risk-neutral equivalent

\[
\max \sum_{t=1}^{T} \mathbb{E}[G_t(s_t, v_t)] + \mathbb{E}[G_{T+1}(s_{T+1})]
\]

\((\mathcal{C})\)

\[s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \ldots, T,\]

\[(G_1(s_1, v_1), \ldots, G_1(s_T, v_T), G_{T+1}(s_{T+1})) \succeq_{\text{dis}}^{(2)} (Y_1, \ldots, Y_T, Y_{T+1})\]

\[v_t \in V_t \text{ a.s., } \quad t = 1, \ldots, T.\]

**Theorem**

If \((\hat{s}, \hat{v})\) is an optimal solution of problem \((\mathcal{C})\), then a random discount sequence \(\xi_t \in \mathcal{L}_\infty(\Omega, \mathcal{F}_t, P), t = 1, \ldots, T + 1\), exists such that \((\hat{s}, \hat{v})\) is an optimal solution of the problem

\[
\max \sum_{t=1}^{T} \mathbb{E}[(1 + \xi_t)G_t(s_t, v_t)] + \mathbb{E}[(1 + \xi_{T+1})G_{T+1}(s_{T+1})]
\]

\[s.t. \quad s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \ldots, T,\]

\[v_t \in V_t \text{ a.s., } \quad t = 1, \ldots, T.\]

**Challenge:** The decisions are not time-consistent.
Two-stage stochastic optimization problems with order constraint on the recourse

First Stage Problem:
\[
\min \limits_x f(x) + \mathbb{E}[\mathcal{Q}(x, \xi)] \\
\text{s.t. } \mathcal{Q}(x, \xi) \preceq_{\text{ic}} Z, \quad x \in \mathcal{D}.
\]

where \( \mathcal{Q}(x, \xi) \) is the optimal value of the second stage problem

Second Stage Problem:
\[
\mathcal{Q}(x, \xi) = \min \limits_y \{ q^\top y : Tx + Wy = h, \ y \in \mathcal{Y} \}.
\]

\( \mathcal{D} \subset \mathbb{R}^n \) and \( \mathcal{Y} \subset \mathbb{R}^m \) are closed convex sets, \( f : \mathbb{R}^n \to \mathbb{R} \) is a convex function, \( \xi = (q, W, T, h) \); \( q, T, h \) are random.

Two-stage stochastic optimization problems with multivariate-order constraint

Two-stage problem:

\[
\min_x f(x) + \mathbb{E}[\mathcal{D}(x, \xi)] \\
\text{s.t. } x \in \mathcal{D}.
\]

\[
\mathcal{D}(x, \xi) = \min_y \{q^\top y : Tx + Wy = h, \ y \in \mathcal{Y} \}
\]

\[
g(y) \succeq Z.
\]

Here \(g : \mathbb{R}^m \to \mathbb{R}^d\) is a continuous mapping and \(Z\) is \(\ell\)-dimensional random vector.

Motivation

**Robotics**: Control of robots’ positions and the communication in a multi-hop fashion within the network and destination centers.

**Portfolio optimization**: Control of return rate and additional performance measures (e.g., drawdown).
Numerical Methods

- **Large scale convex optimization methods** for second order dominance constraints: applicable only to small problems

- **Dual methods** for SSD constraints (DD, Ruszczyński, 2005); (Rudolf, Ruszczyński 2006, Luedtke 2008).

- **Subgradient Based Approximation Methods** for SSD constraints with linear $G(\cdot)$ (Rudolf, Ruszczyński, 2006; Fabian, Mitra, and Roman, 2008)

- **Combinatorial methods** for FSD constraints (Rudolf, Noyan, Ruszczyński 2006) based on second order stochastic dominance relaxation ($\{X : X \succeq^{(2)} Y\} = \overline{co}\{X : X \succeq^{(1)} Y\}$)

- **Methods for two-stage problems with dominance constraints on the recourse** (Schultz, Neise, Gollmer, Drapkin; DD and G. Martinez, 2011)

- **Methods for multivariate linear dominance constraints** (Homem-de-Mello, Mehrotra, 2009; Hu, Homem-de-Mello, and Mehrotra 2010, Armbruster and Luedtke 2010, DD and Wolfhagen 2013)

- **Subgradient methods based on quantile functions and conditional expectations** (DD, Ruszczyński, 2010)

- **Sample average approximation methods** (Sun, Xu and Wang, 2011)
Extensions and Further Research Directions

- **Non-convex problems** Optimality conditions for problems with FSD constraints and problems with higher order dominance constraints with non-convex functions (DD, A Ruszczyński 2004, 2007)

- **Stochastic dominance efficiency in multi-objective optimization** and its relations to dominance constraints (G. Mitra, C. Fabian, K. Darby-Dowman, D. Roman, 2006, 2009)

- **Stability and sensitivity analysis, asymptotic behavior** (DD, R. Henrion, A Ruszczyński, 2007; Y. Liu, H. Xu, 2010; DD and W. Römisch 2011)

- **Semi-infinite composite optimization** (DD, A Ruszczyński 2007)

- **Robust Dominance Relation** (DD, A Ruszczyński, 2010)
Applications

- **Finance**: portfolio optimization
- **Electricity markets**: portfolio of contracts and/or acceptability of contracts
- **Inverse models and forecasting**: Compare the forecast errors via stochastic dominance and design data collection for model calibration
- **Network design**: assign capacity to optimize network throughput
- **Robotics**: control of position and communication of robots
- **Medicine**: radiation therapy designs