Explicit Construction of a Dynamic Bessel Bridge of dimension 3

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Inspired by Kyle (1985), Back (1992) studies a market and its equilibrium price process and cumulative demand for a bond and a risky asset with three types of participants:

1. **Noise traders**: The noise traders have no information about the future value of the risky asset and can only observe their own cumulative demands. The cumulative demand is modeled by a standard Brownian motion denoted with $B$.

2. **Informed trader**: The insider knows the value of the risky asset at time 1, which is given by $f(V)$ where $V$ is a standard normal random variable independent of $B$. Being risk-neutral, her objective is to maximize her expected profit.

3. **Market maker**: The market maker observes the total order and sets the price of the risky asset to clear the market.
Under certain assumptions on the behaviour of market participants, \( X^* \), the equilibrium level of the cumulative demand for the risky asset, satisfies

\[
dX_t^* = dB_t + \frac{V - X_t^*}{1 - t} dt,
\]

so that \( X^* \) is a Brownian bridge. The price of the risky asset in the equilibrium is given by \( H(t, X_t^*) \) for \( H \) solving

\[
H_t + \frac{1}{2}H_{xx} = 0, \quad H(1, x) = f(x).
\]

A key property is that \( X^* \) is a Brownian motion in its own filtration. Thus, when the insider trades optimally she gives the impression that she does not exist to the market.

Back and Pedersen (1998) analyse the same problem when the insider receives a continuous signal

\[ Z_t = Z_0 + \int_0^t \sigma(u) dB_t^Z, \]

where \( Z_0 \sim N(0, c) \), \( B^Z \) is a Brownian motion independent of \( B \), and \( c + \int_0^1 \sigma^2(s) ds = 1 \).

Note that \( Z_t = \beta V(t) \) for some standard Brownian motion \( \beta \) where \( V(t) = c + \int_0^t \sigma^2(s) ds \).

The asset value at time 1 is given by \( Z_1 \). The equilibrium demand in this case is given by

\[ dX_t^* = dB_t + \frac{Z_t - X_t^*}{V(t) - t} dt. \]

Moreover, \( \lim_{t \to 1} X_t^* = Z_1 \), and \( X^* \) is a Brownian motion!

Similar problems in varying generality are discussed in Wu (1999), Föllmer, Wu and Yor (1999) and Danilova (2008).
Extension to a general diffusion setting

Goal: Given

$$Z_t = Z_0 + \int_0^t \sigma(s)a(Z_s)dB_s^Z$$

with $a(z)$ satisfying certain conditions, construct a process $X$, starting from zero and adapted to $\mathcal{F}_t^{Z,B}$, such that:

**C1** For every $T < 1$ $X$ is the strong solution of the SDE

$$dX_t = a(X_t)dB_t + \alpha(t, X_t, Z_t)dt, \quad \text{for } t \in (0, T)$$

for some Borel measurable real valued function $\alpha$.

**C2** $X_1 = Z_1$, $Q^Z$-a.s., where $Q^Z$ is the law of $(X, Z)$ with $Z_0 = z$ and $X_0 = 0$.

**C3** $X$ with $X_0 = 0$ is a local martingale in its own filtration.
In a not so recent paper we have given a positive answer to the previous problem. The solution is given by the solution to the following SDE

\[
dX_t = a(X_t)dB_t + a^2(X_t) \frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)} dt
\]

for \( t < 1 \) for some deterministic function \( \rho \) which will be made precise shortly.

Note that if \( \rho(t, X_t, \cdot) \) is the conditional density of \( Z_t \) given \( \mathcal{F}_t^X \), then \( X \) will be a local martingale in its own filtration.
Indeed, it follows from the standard nonlinear filtering theory that the drift term of $X$ in its own filtration is given by

$$
\mathbb{E} \left[ a^2(X_t) \frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)} \bigg| \mathcal{F}_t^X \right] = a^2(X_t) \frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)} \int \mathcal{R} \rho_x(t, X_t, z) \rho(t, X_t, z) \rho(t, X_t, z) dz = 0
$$

Now, let's assume the initial distribution of $Z$ possesses a density and try to find the conditional distribution of $Z_t$ for the given drift term.
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$$

$$
= a^2(X_t) \frac{\partial}{\partial x} \int_{\mathbb{R}} \rho(t, X_t, z) \, dz
$$

$$
= 0
$$
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$$

Now, let’s assume the initial distribution of $Z$ possesses a density and try to find the conditional distribution of $Z_t$ for the given drift term.
It follows from nonlinear filtering theory that the conditional density, \( g \), of \( Z_t \) satisfies the SPDE

\[
g_t(z) = P(Z_0 \in dz)/dz + \int_0^t \frac{1}{2} \sigma^2(s) (a^2(z)g_s(z))_{zz} ds \\
+ \int_0^t g_s(z) \left( \frac{\rho_x(s, X_s, z)}{\rho(s, X_s, Z_s)} - \int_{\mathbb{R}} g_s(z) \frac{\rho_x(s, X_s, z)}{\rho(s, X_s, Z_s)} dz \right) dN_s,
\]

where \( dN_s = dX_s - \left( \int_{\mathbb{R}} g_s(z) \frac{\rho_x(s, X_s, z)}{\rho(s, X_s, Z_s)} dz \right) ds \).

If \( X \) is a martingale in its own filtration, the above turns into

\[
g_t(z) = P(Z_0 \in dz)/dz + \int_0^t \frac{1}{2} \sigma^2(s) (a^2(z)g_s(z))_{zz} ds \\
+ \int_0^t g_s(z) \frac{\rho_x(s, X_s, z)}{\rho(s, X_s, Z_s)} dX_s.
\]
It follows from an application of Ito’s formula that in order for $\rho$ to give the conditional density it must be a solution to the PDE

$$\rho_t + \frac{1}{2} a^2(x)\rho_{xx} - \frac{1}{2} \sigma^2(t)(a^2(z)\rho)_{zz} = 0 \quad (3)$$

with the initial condition that $\rho(0, 0, z) \, dz = P(Z_0 \in dz)$.

Conversely, if one can solve the above PDE, the solution also solves the SPDE in (1). Moreover, $X$ is a local martingale in its own filtration.

It might seem hopeless to solve this PDE and in general there may not be any solution for such PDEs.
Let $G$ be the transition density of a diffusion solving

$$d\xi_t = a(\xi_t) d\beta_t,$$  

where $\beta$ is a standard Brownian motion. I.e. $G(t, x, z) dz = P(\xi_{s+t} \in dz | \xi_s = x)$.

We shall now see that $\rho(t, x, z) = G(V(t) - t, x, z)$ solves the PDE above as soon as we assume $P(Z_0 \in dz) = G(c, 0, z)$, where $c = V(0)$. 

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where $c = V(0)$.

First recall that, being the transition density of $\xi$, $G(t - s, x, z)$ is the fundamental solution of

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First recall that, being the transition density of $\xi$, $G(t - s, x, z)$ is the fundamental solution of

$$u_t = \frac{1}{2}(a^2(z)u)_{zz},$$

and also of the adjoint equation

$$v_s + \frac{1}{2}a^2(x)v_{xx} = 0.$$
These two facts imply that $G(V(t) - t, x, z)$ is a solution to (3).

Thus, if we choose $\rho(t, x, z) = G(t, x; V(t), z)$, then $X$ will be a local martingale in its own filtration.
These two facts imply that $G(V(t) - t, x, z)$ is a solution to (3).

Thus, if we choose $\rho(t, x, z) = G(t, x; V(t), z)$, then $X$ will be a local martingale in its own filtration.

Note that since $V(1) = 1$, as $t \to 1$, $\rho$ converges to the delta function at $x$. This will in turn imply that $X_t \to Z_1$ as $t \to 1$.

It is interesting to observe that placing $X$ as a backward variable and $Z$ as a forward variable gives a martingale that converges to the terminal value of $Z$. 
Let’s assume that the “firm value” is given by a standard Brownian motion \( \beta \) with \( \beta_0 = 1 \), and the insider observes \( Z_t := \beta V(t) \). Suppose \( V(0) = 0 \). Thus, we can write

\[
Z_t = 1 + \int_0^t \sigma(s) dB^Z_s
\]

for some Brownian motion \( B^Z \).

Let the default time be the first time that \( \beta \) becomes 0 so that it equals \( V(\tau) \) where \( \tau := \inf\{t > 0 : Z_t = 0\} \).

If one tries to find the conditions for equilibrium, in particular in view of the results of Campi and Çetin, it seems that in the equilibrium, the insider should try to drive the total demand plus 1, \( X \), to 0 at \( V(\tau) \) and should not allow \( X \) hit 0 before \( V(\tau) \).

Moreover, the insider will try to be insconspicuous, i.e. \( X \) is a Brownian motion.
From now on we shall consider the problem in infinite horizon, i.e. \( t \in [0, \infty) \). Our assumption on \( V \) is \( V(t) > t \) for all \( t > 0 \) and \( V(0) = 0 \).

Let

\[
q(t, x, y) := \frac{1}{\sqrt{2\pi t}} \left( \exp \left( -\frac{(x - y)^2}{2t} \right) - \exp \left( -\frac{(x + y)^2}{2t} \right) \right),
\]

for \( x > 0 \) and \( y > 0 \) be the transition density of a standard Brownian motion killed at 0. Recall that

\[
H(t, a) := \mathbb{P} [ T_a > t ] = \int_t^\infty \ell(u, a) \, du = \int_0^\infty q(t, a, y) \, dy \quad (5)
\]

for \( a > 0 \) where

\[
T_a := \inf \{ t > 0 : B_t = a \}, \text{ and } \\
\ell(t, a) := \frac{a}{\sqrt{2\pi t^3}} \exp \left( -\frac{a^2}{2t} \right).
\]
Our standing assumption is that there exists some $\varepsilon > 0$ such that $\int_0^\varepsilon \frac{1}{(V(t)-t)^2} \, dt < \infty$.

**Theorem 1**

*There exists a unique strong solution to*

$$X_t = 1 + B_t + \int_0^{\tau \wedge t} \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)} \, ds$$

$$+ \int_{\tau \wedge t}^{V(\tau) \wedge t} \frac{\ell_a(V(\tau) - s, X_s)}{\ell(V(\tau) - s, X_s)} \, ds$$

*such that*

i) $X$ is a standard Brownian motion in its own filtration, and

ii) $V(\tau) = \inf\{ t > 0 : X_t = 0 \}$. 

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Our goal is to construct a continuous process which solves

\[ dX_t = dB_t + \alpha(t, X_t, Z_t)dt, \]

with \( X_0 = 1 \) for some drift \( \alpha \) such that \( X \) is a Brownian motion in its own filtration and \( V(\tau) = \inf\{ t > 0 : X_t = 0 \} \).
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The drift after \( \tau \) is that of a 3-dimensional Bessel bridge, and this choice follows from the relationship between the distribution of the Bessel bridge and that of a Brownian motion when it is conditioned on its first hitting time of 0.
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The drift after \( \tau \) is that of a 3-dimensional Bessel bridge, and this choice follows from the relationship between the distribution of the Bessel bridge and that of a Brownian motion when it is conditioned on its first hitting time of 0.

In order to guess the right choice for \( \alpha \) until \( \tau \) we have used the following absolutely continuous measure change. Define \( Q_t \) on \( (\Omega, \mathcal{H}_t) \) by

\[ Q_t(E) = \frac{\mathbb{P}[\tau > t, E]}{\mathbb{P}[\tau > t]}, \]

for any \( E \in \mathcal{H}_t \). \( Q_t \) is absolutely continuous with respect to the restriction of \( \mathbb{P} \) to \( \mathcal{H}_t \), which we will denote with \( \mathbb{P}_t \).
An $h$-transform

Consider the martingale $M$ defined by, for $s \leq t$

$$M_s = E \left[ \frac{dQ_t}{dP_t} | \mathcal{H}_s \right] = \frac{P[\tau > t | \mathcal{H}_s]}{P[\tau > t]}$$
Consider the martingale $M$ defined by, for $s \leq t$

$$M_s = \mathbb{E} \left[ \frac{dQ_t}{dP_t} \mid \mathcal{H}_s \right] = \frac{\mathbb{P}[\tau > t \mid \mathcal{H}_s]}{\mathbb{P}[\tau > t]}$$

$$= 1_{[\tau > s]} \frac{H(V(t) - V(s), Z_s)}{H(V(t), 1)},$$

where $H$ was defined by (5) since $Z$ is merely a time-changed Brownian motion where the time change is deterministic.
Consider the martingale $M$ defined by, for $s \leq t$

$$M_s = \mathbb{E} \left[ \frac{d\mathbb{Q}_t}{d\mathbb{P}_t} \bigg| \mathcal{H}_s \right] = \frac{\mathbb{P}[\tau > t | \mathcal{H}_s]}{\mathbb{P}[\tau > t]}$$

$$= \mathbf{1}_{[\tau > s]} \frac{H(V(t) - V(s), Z_s)}{H(V(t), 1)},$$

where $H$ was defined by (5) since $Z$ is merely a time-changed Brownian motion where the time change is deterministic.

A straightforward application of Girsanov theorem yields that under $\mathbb{Q}_t$, $Z$ has the following dynamics

$$dZ_s = \sigma(s) d\tilde{B}_s + \sigma^2(s) \frac{H_x(V(t) - V(s), Z_s)}{H(V(t) - V(s), Z_s)} ds,$$

for $s \leq t$ for some $\mathbb{Q}_t$-Brownian motion $\tilde{B}$. Note that $\mathbb{Q}_t[\tau > t] = 1$. 

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Under this h-transform the transition density of $Z$ is given by

$$Q_t[Z_s \in dz | Z_r = X] = p(V(t); V(r), V(s), x, z)$$

where

$$p(t; r, s, x, z) = q(s - r, x, z) \frac{H(t - s, z)}{H(t - r, x)}.$$  \hspace{1cm} (7)

Note that $p$ is the transition density of the Brownian motion killed at 0 after the analogous h-transform where the h-function is given by $H(t - s, x)$.

Recall that we want $X$ to hit 0 for the first time at $V(\tau)$. If we denote this hitting time with $T$ then we immediately have $[\tau > t] = [T > V(t)]$ due to the monotonicity of $V$. In view of the preceding remarks for $Z$ and the killed Brownian motion we expect $X$ has the following dynamics under $Q_t$ and with respect to its own filtration:

$$dX_t = dB_t^X + \frac{H_x(V(t) - s, X_s)}{H(V(t) - s, X_s)} ds.$$
We look for a pair \((Z, X)\) that, under \(\mathbb{Q}_t\) and with respect to the filtration \((\mathcal{H}_t)\), jointly solve the following system:

\[
\begin{align*}
dZ_s &= \sigma(s)d\tilde{B}_s + \sigma^2(s) \frac{H_x(V(t) - V(s), Z_s)}{H(V(t) - V(s), Z_s)} ds \\
dX_s &= dB_s + \left\{ \frac{H_x(V(t) - s, X_s)}{H(V(t) - s, X_s)} + \right. \\
&\quad \left. \frac{H_x(V(t) - V(s), Z_s)}{H(V(t) - V(s), Z_s)} \right\} ds
\end{align*}
\]
Candidate drift

We look for a pair \((Z, X)\) that, under \(\mathbb{Q}_t\) and with respect to the filtration \((\mathcal{H}_t)\), jointly solve the following system:

\[
\begin{align*}
dZ_s &= \sigma(s)d\tilde{B}_s + \sigma^2(s)\frac{H_x(V(t) - V(s), Z_s)}{H(V(t) - V(s), Z_s)} ds \\
dX_s &= dB_s + \left\{ \frac{H_x(V(t) - s, X_s)}{H(V(t) - s, X_s)} + \frac{\rho_x(t; s, X_s, Z_s)}{\rho(t; s, X_s, Z_s)} \right\} ds
\end{align*}
\]

for some function \(\rho(t; s, x, z)\) which we expect to be the \(\mathbb{Q}_t\) conditional density of \(Z_s\) given \(\mathcal{F}_s^X\) and \(X_s = x\).
Candidate drift

We look for a pair \((Z, X)\) that, under \(\mathbb{Q}_t\) and with respect to the filtration \((\mathcal{H}_t)\), jointly solve the following system:

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\begin{align*}
  dZ_s &= \sigma(s) d\tilde{B}_s + \sigma^2(s) \frac{H_x(V(t) - V(s), Z_s)}{H(V(t) - V(s), Z_s)} ds \\
  dX_s &= dB_s + \left\{ \frac{H_x(V(t) - s, X_s)}{H(V(t) - s, X_s)} + \frac{\rho_x(t; s, X_s, Z_s)}{\rho(t; s, X_s, Z_s)} \right\} ds
\end{align*}
\]

for some function \(\rho(t; s, x, z)\) which we expect to be the \(\mathbb{Q}_t\) conditional density of \(Z_s\) given \(\mathcal{F}_s^X\) and \(X_s = x\).

Doing similar computations as we did before, \(\rho\) will be the desired conditional density if it solves, for fixed \(t\),

\[
\rho_s + \frac{H_x(V(t) - s, x)}{H(V(t) - s, x)} \rho_x + \frac{1}{2} \rho_{xx} = -\sigma^2(s) \left( \frac{H_x(V(t) - V(s), z)}{H(V(t) - V(s), z)} \rho \right)_z + \frac{1}{2} \sigma^2(s) \rho_{zz}.
\]

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The LHS of (8) is the backward equation and the RHS is the forward equation associated with the Markov process with the transition density given by (7). Then, it can be shown similarly to the earlier filtering problem that the solution to (8) is given by

$$
\rho(t; s, x, z) = q(V(s) - s, x, z) \frac{H(V(t) - V(s), z)}{H(V(t) - s, x)}.
$$
The LHS of (8) is the backward equation and the RHS is the forward equation associated with the Markov process with the transition density given by (7). Then, it can be shown similarly to the earlier filtering problem that the solution to (8) is given by

\[ \rho(t; s, x, z) = q(V(s) - s, x, z) \frac{H(V(t) - V(s), z)}{H(V(t) - s, x)}. \]

This would imply that under \( \mathbb{Q}_t \)

\[ dX_s = dB_s + \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)} ds \quad (9) \]

for \( s \leq t \).
Recall that
\[ \frac{\ell_a}{\ell}(t, a) = \frac{1}{a} - \frac{a}{t}, \]
and the drift term after \( \tau \) is that of a 3-dimensional Bessel bridge of length \( V(\tau) \) conditioned to hit 0 at \( V(\tau) \).

Thus, if one can show that \( X \) is strictly positive over \([0, \tau]\), this will imply \( V(\tau) \) is the first time that \( X \) hits 0.

We show this by constructing a weak solution over \([0, \tau]\) which is strictly positive and showing that pathwise uniqueness holds in this interval.
On strict positivity

Let

\[ Y_t = y + B_t + \int_0^\tau q_x (V(s) - s, Y_s, Z_s) ds \quad y > 0. \]

The right monotonicity of \( q_x (t, x, z) \) in \( x \) gives the pathwise uniqueness of the above SDE. Thus, if one obtains a weak solution that is strictly positive, we are done since pathwise uniqueness plus a weak solution implies existence of a strong solution.
On strict positivity

Let

\[ Y_t = y + B_t + \int_0^{\tau \wedge t} \frac{q_x}{q} (V(s) - s, Y_s, Z_s) \, ds \quad y > 0. \]

- The right monotonicity of \( \frac{q_x}{q} (t, x, z) \) in \( x \) gives the pathwise uniqueness of the above SDE. Thus, if one obtains a weak solution that is strictly positive, we are done since pathwise uniqueness + a weak solution implies existence of a strong solution.

- A positive weak solution can be constructed via a Girsanov transformation, details can be found in the paper.
Let

\[ Y_t = y + B_t + \int_0^{\tau \wedge t} \frac{q_x}{q}(V(s) - s, Y_s, Z_s) \, ds \quad y > 0. \]

The right monotonicity of \( \frac{q_x}{q}(t, x, z) \) in \( x \) gives the pathwise uniqueness of the above SDE. Thus, if one obtains a weak solution that is strictly positive, we are done since pathwise uniqueness+a weak solution implies existence of a strong solution.

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One can get a tighter lower bound: Note that

\[ \frac{q_x}{q}(t, x, 0) = \frac{1}{x} - \frac{x}{t}, \]
Next consider the SDE

\[ R_t = y + B_t + \int_0^t \left\{ \frac{1}{R_s} - \frac{R_s}{V(s) - s} \right\} ds. \]
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$$R_t = y + B_t + \int_0^t \left\{ \frac{1}{R_s} - \frac{R_s}{V(s) - s} \right\} ds.$$ 

$R$ can be shown to be a scaled and time-changed 3-dimensional Bessel process, i.e. $R_t = \lambda_t r_{\Lambda_t}$ where $r$ is a 3-dimensional Bessel process and

$$\lambda_t := \exp \left( - \int_0^t \frac{1}{V(s) - s} ds \right),$$

$$\Lambda_t := \int_0^t \frac{1}{\lambda_s^2} ds.$$

It follows from Tanaka’s formula, our standing assumption on $V$ and Gronwall inequality that $(R - Y)^+ \equiv 0.$
Let $\tilde{G}_t := \sigma(\{X_s, s \leq t\}, \tau \wedge t)$ and $G_t = \mathcal{N} \bigvee_{u > t} \tilde{G}_u$, where $\mathcal{N}$ is the set of $\mathbb{P}$-null sets, so that $\mathcal{G} = (G_t)_{t \geq 0}$ is a filtration satisfying usual conditions.

**Proposition 1**

Let $X$ be the unique strong solution of (6) and $f : \mathbb{R}^+ \mapsto \mathbb{R}$ be a bounded measurable function with a compact support contained in $(0, \infty)$.

$$
\mathbb{E}[\mathbb{1}_{[\tau > t]} f(Z_t) | \mathcal{G}_t] = \mathbb{1}_{[\tau > t]} \int_0^\infty f(z) \frac{q(V(t) - t, X_t, z)}{H(V(t) - t, X_t)} \, dz.
$$
Using the above proposition we can easily obtain the $G$-canonical decomposition of $X$.

**Corollary 2**

Let $X$ be the unique strong solution of (6). Then,

$$M_t := X_t - 1 - \int_0^{\tau \wedge t} \frac{H_x(V(s) - s, X_s)}{H(V(s) - s, X_s)} \, ds - \int_{\tau \wedge t}^{V(\tau) \wedge t} \frac{\ell_a(V(\tau) - s, X_s)}{\ell(V(\tau) - s, X_s)} \, ds$$

is a standard $G$-Brownian motion vanishing at 0.

One way to show that $X$ is a Brownian motion is to project the above representation onto $\mathbb{F}^X$, the minimal filtration generated by $X$ satisfying the usual conditions. However, there is an easier way. We shall next find the canonical decomposition of $X$ under $G^\tau := (\mathcal{G}_t^\tau)_{t \geq 0}$ where $\mathcal{G}_t^\tau = \mathcal{G}_t \vee \sigma(\tau)$. Note that $\mathcal{G}_t^\tau = \mathcal{F}_t^X \vee \sigma(\tau)$.
Using the conditional distribution of $Z$ we obtain
\[
\mathbb{P}[\tau \in du, \tau > t | G_t] = 1_{[\tau > t]} \sigma^2(u) \frac{\ell(V(u) - t, X_t)}{H(V(t) - t, X_t)} du,
\]
for $u > t$. Thus, using the well known results in the theory of enlargement of filtrations,
\[
X_t - 1 - \int_0^{V(\tau) \wedge t} \frac{\ell_a(V(\tau) - s, X_s)}{\ell(V(\tau) - s, X_s)} ds
\]
is a Brownian motion. Thus, $X$ in its own filtration enlarged with its first hitting time of 0 has the same decomposition of a Brownian motion in its own filtration enlarged with its first hitting time of 0. Thus, $X$ has the same law as a standard Brownian motion since $V(\tau) = T_1$ in distribution.