On new advancements in Robust Pricing and Hedging

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parts of this talk are based on joint works with

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Outline

Modelling in Mathematical Finance
   Classical modelling framework
   Towards a Robust modelling framework

Theory of robust valuation and hedging (duality)
   Abstract no-duality gap results
   Explicit solutions via SEP: one and $n$ marginals

Beliefs based on historic time series
   Confidence intervals for realised volatility
   Robust Market Models

A class of diffusion models and their static hedging

Robust hedging in practice
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Modelling: Classical approach

The prevailing approach in math finance proceeds as follows:

- Write down a plausible and well behaved stochastic model.
- Compute prices of (liquid) financial instruments as function of model parameters.
- Calibrate the model: chose the parameters to match the prices already observed in the market.
- Use it: sell and hedge derivatives.
In the wake of financial crisis its critique has been vocal...

the economics profession went astray because economists, as a group, mistook beauty, clad in impressive-looking mathematics, for truth (Paul Krugman ’09)

Misplaced reliance on sophisticated maths [...] inherent ‘Knightian’ uncertainty. (The Turner Review ’09)

we need insights into the omnipresent model risk. [...] the program of introducing a Robust framework is one the greatest challenges also from the mathematical perspective (Hans Föllmer ’09)

For banks, the only way to avoid a repetition of the current crisis is to measure and control all their risks, including the risk that their models give incorrect results (Steven Shreve ’08)

the problem is that we don’t have enough math. [...] Frictions are just bloody hard with the mathematical tools we have now (John Cochrane ’09)
Modelling: Key Ingredients

Inputs:
- Beliefs (about the future dynamics of the stock price process)
- Information (market prices)
- Rules (self-financing trading strategies, frictions)

Reasoning principles:
- Markets are efficient – there are *no arbitrages*.

Outputs:
- Prices and hedges of liabilities
- Portofolio optimisation
- Risk management
Modelling: Classical approach

INPUTS:

- **Beliefs** (about the future dynamics of the stock price process):
  
  Prices of risky assets \( S_t^i \) have given dynamics and are semimartingales on a fixed \( \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P} \).

- **Information** (market prices): \( S_0^i \)
  
  Market quotes used to calibrate the above by choosing free parameters via reverse engineering.

- **Rules** (self-financing trading strategies, frictions):
  
  \( s-f \) trading strategy = stochastic integral; no frictions

REASONING PRINCIPLES:

- **Markets are efficient – there are no arbitrages**:
  
  Fundamental theorem of asset pricing (FTAP) asserts that no-arbitrage is equivalent to existence of \( \mathbb{Q} \) equivalent to \( \mathbb{P} \) under which (discounted) \( S_t^i \) are all martingales.

OUTPUTS:

- **Prices and hedges of liabilities** – via cost of replication; computing expectations of functionals of solutions to SDEs
- **Portfolio optimisation** – stochastic control theory
- **Risk management** – duality, convex and functional analysis
Modelling: Classical approach

Inputs:

- **Beliefs** (about the future dynamics of the stock price process): 
  *Prices of risky assets ($S_t^i$) have given dynamics and are semimartingales on a fixed $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.**
- **Information** (market prices): $S_0$
  Market quotes used to calibrate the above by choosing free parameters via reverse engineering.
- **Rules** (self-financing trading strategies, frictions):
  *s-f trading strategy = stochastic integral; no frictions*

Reasoning principles:

- **Markets are efficient – there are no arbitragers:**
  *Fundamental theorem of asset pricing (FTAP) asserts that no-arbitrage is equivalent to existence of $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which (discounted) $S_t^i$ are all martingales.*

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- **Information** (market prices): \( S_0^i \)
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**Outputs:**

- Prices and hedges of liabilities – via cost of replication;
  *computing expectations of functionals of solutions to SDEs*

- Portofolio optimisation – *stochastic control theory*

- Risk management – *duality, convex and functional analysis*
Recently, its critique has been vocal ...

The drawbacks of the classical approach include:

- It is exposed to model uncertainty which may be hard to quantify.
- Inevitably, it ignores information present in the market. Models are re-calibrated daily: theoretically inconsistent.
- It is a first order approximation which ignores market frictions.

A robust approach aims to address these weaknesses. It does not replace the classical framework but offers an alternative allowing for a comprehensive understanding and control of risk.
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The drawbacks of the classical approach include:

- It is exposed to model uncertainty which may be hard to quantify.
- Inevitably, it ignores information present in the market. Models are re-calibrated daily: theoretically inconsistent.
- It is a first order approximation which ignores market frictions.

A robust approach aims to address these weaknesses. It does not replace the classical framework but offers an alternative allowing for a comprehensive understanding and control of risk.
Relaxing Beliefs: Model uncertainty

Inputs:

- **Beliefs** (about the future dynamics of the stock price process): 
  *Pricing of the assets* \( (S_t^i) \) *are adapted to* \((\Omega, \mathcal{F}, (\mathcal{F}_t))\) *and we have a set of possible measures* \((\mathbb{P}_\alpha)\).
- **Information** (market prices): only \( S_0^i \).
- **Rules** (self-financing trading strategies, frictions): 
  *no frictions; trading strategy = stochastic integral: classical and its extensions (quasi-sure analysis, G-expectation...*)

Reasoning principles:

- Markets are efficient – there are no arbitrages: 
  *Essentially link back to the classical FTAP.*

Outputs:

- Robust Portofolio optimisation and Risk management
- Price bounds and super/sub-hedges of liabilities

see e.g. Rogers (01), Maccheroni et al. (06), Schied (07); Lyons (95), Avellaneda et al. (95), Denis and Martini (06), Peng (07), Soner et al. (11), Bouchard and Nutz (13); Bick and Willinger (94), Cassese (08), Vovk (12)
Including Information: Market Models

**Inputs:**

- **Beliefs** (about the future dynamics of the stock price process):
  A probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\) is fixed and we specify the joint dynamics of the asset \(S_t\) and the option prices \(P^{K,T}_t\).

- **Information** (market prices):
  full use – market prices simply specify the initial values of \(S_0, P^K_0, T\).

- **Rules** (self-financing trading strategies, frictions):
  no frictions; trading strategies are stochastic integrals.

**Reasoning principles:**

- Markets are efficient – there are no arbitrages:
  The classical FTAP.

**Outputs:**

- In principle all prices/hedges are specified clearly if one can compute them.

see Schonbucher (99), Bergomi (05), Schweizer and Wissel (08), Jacod and Protter (10), Carmona and Natochiy (09,11),
**B & I: the Robust Approach** (Cox & O. (11))

**Inputs:**
- **Beliefs** (about the future dynamics of the stock price process):
  \[ \text{Prices of risky assets } (S_t^i)_{t \leq T} \text{ belong to some path space } \mathcal{P}. \]
- **Information** (market prices): set of payoffs \( \mathcal{X} \),
  \[ \mathcal{X} : \mathcal{P} \rightarrow \mathbb{R}, \text{ with given prices } \mathcal{P} : \mathcal{X} \rightarrow \mathbb{R} \]
- **Rules** (self-financing trading strategies, frictions): \textit{no frictions, trading } \sim \textit{ simple trading; pathwise stochastic integrals}

**Reasoning principles:**
- **New FTAP needed:**
  \[ \text{no-arbitrage } \iff \text{ exists a } (\mathcal{P}, \mathcal{P}, \mathcal{X})-\text{market model i.e. a classical setup } (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{Q}, (S_t)) \text{ with } (S_t) \in \mathcal{P} \text{ a.s., } S_t \text{ a } \mathcal{Q}-\text{martingale and with } \mathcal{P} \mathcal{X} = E^\mathcal{Q} X, X \in \mathcal{X}. \]
  \[ \text{no-arbitrage } \iff \text{ restrictions on } \mathcal{P} \text{ and on } \mathcal{P}. \]

**Outputs:** Consider an option \( O_T : \mathcal{P} \rightarrow \mathbb{R} \) and investigate
\[ \mathcal{P} \text{ admits no arbitrage on } \mathcal{X} \cup \{ O_T \} \iff \text{LB } \leq \mathcal{P} O_T \leq \text{UB} \]
\[ \iff \text{ there exists a } (\mathcal{P}, \mathcal{P}, \mathcal{X} \cup \{ O_T \})-\text{market model} \]
& derive robust hedging strategies which enforce LB and UB.
B & I: the Robust Approach (Cox & O. (11))

**Inputs:**
- **Beliefs** (about the future dynamics of the stock price process): Prices of risky assets \((S^i_t)_{t \leq T}\) belong to some path space \(\mathcal{P}\).
- **Information** (market prices): set of payoffs \(\mathcal{X}\), \(\mathcal{X} \ni X : \mathcal{P} \rightarrow \mathbb{R}\), with given prices \(\mathcal{P} : \mathcal{X} \rightarrow \mathbb{R}\).
- **Rules** (self-financing trading strategies, frictions): no frictions, \(\text{trading } \rightsquigarrow \text{simple trading; pathwise stochastic integrals}\).

**Reasoning principles:**
- **New FTAP needed:**
  - no-arbitrage \(\iff\) exists a \((\mathcal{P}, \mathcal{P}, \mathcal{X})\)-market model i.e. a classical setup \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{Q}, (S_t))\) with \((S_t) \in \mathcal{P} a.s., S_t a \mathcal{Q}\text{-martingale and with } \mathcal{P}X = \mathbb{E}_\mathcal{Q}X, X \in \mathcal{X}\).
  - no-arbitrage \(\iff\) restrictions on \(\mathcal{P}\) and on \(\mathcal{P}\).

**Outputs:** Consider an option \(O_T : \mathcal{P} \rightarrow \mathbb{R}\) and investigate
- \(\mathcal{P}\) admits no arbitrage on \(\mathcal{X} \cup \{O_T\}\) \(\iff\) \(LB \leq \mathcal{P}O_T \leq UB\)
- \(\iff\) there exists a \((\mathcal{P}, \mathcal{P}, \mathcal{X} \cup \{O_T\})\)-market model & derive robust hedging strategies which enforce \(LB\) and \(UB\).
B & I: the Robust Approach (Cox & O. (11))

Inputs:

- **Beliefs** (about the future dynamics of the stock price process): 
  \( \text{Prices of risky assets } (S_t^i)_{t \leq T} \text{ belong to some path space } \mathcal{P}. \)
- **Information** (market prices): set of payoffs \( \mathcal{X} \), 
  \( \mathcal{X} \ni X : \mathcal{P} \to \mathbb{R} \), with given prices \( \mathcal{P} : \mathcal{X} \to \mathbb{R} \)
  \( \text{trading } \rightsquigarrow \text{ simple trading}; \textit{pathwise stochastic integrals} \)

Reasoning principles:

- **New FTAP needed:**
  \( \text{no-arbitrage } \iff \text{exists a } (\mathcal{P}, \mathcal{P}, \mathcal{X})-\text{market model } \text{i.e. a} \)
  \( \text{classical setup } (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{Q}, (S_t)) \text{ with } (S_t) \in \mathcal{P} \text{ a.s., } S_t \text{ a} \)
  \( \text{Q-martingale and with } \mathcal{P}X = \mathbb{E}^\mathcal{Q}X, X \in \mathcal{X}. \)
  \( \text{no-arbitrage } \iff \text{restrictions on } \mathcal{P} \text{ and on } \mathcal{P}. \)

Outputs: Consider an option \( O_T : \mathcal{P} \to \mathbb{R} \) and investigate
\( \mathcal{P} \text{ admits no arbitrage on } \mathcal{X} \cup \{O_T\} \iff \text{LB } \leq \mathcal{P}O_T \leq \text{UB} \)
\( \iff \text{there exists a } (\mathcal{P}, \mathcal{P}, \mathcal{X} \cup \{O_T\})-\text{market model} \)
\& derive \textit{robust hedging} strategies which enforce \text{LB} and \text{UB}. 

Robust approach to Mathematical Finance

Ann Arbor, Sep 2013

Jan Obloj
B & I: the Robust Approach (Cox & O. (11))

**Outputs:** Consider an option \( O_T : \mathcal{P} \rightarrow \mathbb{R} \) and investigate

\[ \mathcal{P} \text{ admits no arbitrage on } \mathcal{X} \cup \{ O_T \} \iff LB \leq \mathcal{P} O_T \leq UB \]

\[ \iff \text{there exists a } (\mathcal{P}, \mathcal{P}, \mathcal{X} \cup \{ O_T \})-\text{market model} \]

& derive robust hedging strategies which enforce \( LB \) and \( UB \).

Such no-arbitrage bounds and super-/sub- hedges obtained in:

- Hobson (98); Brown, Hobson and Rogers (01), Dupire (05), Lee (07),
- Cox, Hobson and O. (08), Cox and Wang (12); cf. O. (10), Hobson (10).
- Davis and Hobson (07), Cox and O. (11,11) and Davis, O. and Raval (12) also look at no-arbitrage questions.

Duality for expressing \( LB \) & \( UB \) in general considered in recent works:

- Davis, O. and Raval; Galichon, Henry-Labordère and Touzi; Beiglböck,
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Discrete time models

**Inputs:**
- **Beliefs:** $\mathcal{P} = \mathbb{R}_+^n$
- **Information:** set of payoffs $\mathcal{X} = \{(S_t - K)^+ : K \in \mathbb{K}_t, \ t = 1, \ldots, n\}$ with given prices.
- **Rules:** no frictions, discrete time trading in $S$, static in calls.

**Reasoning principles:** Assume $\mathcal{M}_\mathcal{P} \neq \emptyset$ and no strong arbitrage.

**Outputs:** the superhedging duality holds:

$$\sup_{\mathcal{M}_\mathcal{P}} (\text{payoff expectation}) = \inf_{\mathcal{P}} \{\mathcal{P}(\text{superhedge})\}$$
Discrete time models

**INPUTS:**

- **Beliefs:** \( \mathcal{P} = \mathbb{R}_+^n \)
- **Information:** set of payoffs \( \mathcal{X} = \{(S_t - K)^+ : K \in \mathbb{K}_t, \ t = 1, \ldots, n\} \) with given prices.
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**REASONING PRINCIPLES:** Assume \( \mathcal{M}_\mathcal{P} \neq \emptyset \) and *no strong arbitrage*.

**OUTPUTS:** the superhedging duality holds:

\[
\sup_{\mathcal{M}_\mathcal{P}} (\text{payoff expectation}) = \inf_{\mathcal{M}_\mathcal{P}} \{\mathcal{P}(\text{superhedge})\}
\]

Davis, O. and Raval (13): \( n = 1, \mathbb{K}_1 \) a finite set. Boils down to linear programming.
Discrete time models

**Inputs:**
- **Beliefs:** $\mathcal{P} = \mathbb{R}^n_+$
- **Information:** set of payoffs $\mathcal{X} = \{ (S_t - K)^+ : K \in \mathbb{K}_t, \ t = 1, \ldots, n \}$ with given prices.
- **Rules:** *no frictions*, discrete time trading in $S$, static in calls.

**Reasoning principles:** Assume $\mathcal{M}_\mathcal{P} \neq \emptyset$ and *no strong arbitrage*.

**Outputs:** the superhedging duality holds:

$$\sup_{\mathcal{M}_\mathcal{P}} \text{(payoff expectation)} = \inf \{ \mathcal{P}(\text{superhedge}) \}$$

Beiglböck, Henry-Labordère and Penkner (13): arbitrary $n$, $\mathbb{K}_i = \mathbb{R}_+$.

Develop Moge-Kantorovich (martingale) mass transport approach.
Discrete time models

**INPUTS:**

- **Beliefs:** \( \mathcal{P} = \mathbb{R}_+^n \)
- **Information:** set of payoffs  
  \[ \mathcal{X} = \{(S_t - K)^+ : K \in \mathbb{K}_t, \ t = 1, \ldots, n\} \] with given prices.
- **Rules:** *no frictions*, discrete time trading in \( S \), static in calls.

**REASONING PRINCIPLES:** Assume \( \mathcal{MP} \neq \emptyset \) and *no strong arbitrage*.

**OUTPUTS:** the superhedging duality holds:

\[
\sup_{\mathcal{MP}} (\text{payoff expectation}) = \inf_{\mathcal{MP}} \{\mathcal{P}(\text{superhedge})\}
\]

Acciaio, Beiglböck, Penkner and Schachermayer (13): arbitrary \( n \) and general \( \mathcal{X} \) (with technical assumptions); also show a robust FTAP.

Bouchard and Nutz (13): finite \( \mathcal{X} \).
Continuous time models

**Inputs:**
- **Beliefs:** $\mathcal{P} = C([0, T])$
- **Information:** call options $\mathcal{X} = \{(S_T - K)^+ : K \geq 0\}$ with given prices.
- **Rules:** *no frictions*, finite variation trading in $S$, static in calls.

**Reasoning principles:** Assume $\mathcal{M}_P \neq \emptyset$ and no strong arbitrage.

**Outputs:** Dolinsky and Soner (13), for Lipschitz continuous payoffs and under technical assumptions, show that duality holds.
Continuous time models

**Inputs:**

- **Beliefs:** $\mathcal{P} = C([0, T])$
- **Information:** call options $\mathcal{X} = \{(S_T - K)^+ : K \geq 0\}$ with given prices.
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**Reasoning principles:** Assume $\mathcal{M}_\mathcal{P} \neq \emptyset$ and no strong arbitrage.

**Outputs:** Dolinsky and Soner (13), for Lipschitz continuous payoffs and under technical assumptions, show that duality holds:

$$\sup_{\mathcal{M}_\mathcal{P}} (\text{payoff expectation}) = \inf \{ \mathcal{P}(\text{superhedge}) \}$$
Continuous time models

**INPUTS:**

- **Beliefs:** $\mathcal{P}$ a closed subset of $C([0, T])$
- **Information:** call options $\mathcal{X} = \{(S_{T_i} - K)^+ : K \geq 0\}$, $i = 1, \ldots, n$ with given prices.
- **Rules:** *no frictions*, finite variation trading in $S$, static in calls.

**REASONING PRINCIPLES:** Assume $\mathcal{M}_\mathcal{P} \neq \emptyset$ and no strong arbitrage.

**OUTPUTS:** with Hou, for Lipschitz continuous payoffs and under technical assumptions, we show that duality holds:

$$\sup_{\mathcal{M}_\mathcal{P}} (\text{payoff expectation}) = \inf_{\mathcal{P}} \{\mathcal{P}(\text{superhedge on } \mathcal{P})\}$$
Continuous time models – SEP & one marginal

- **Beliefs:** $\mathfrak{P} = C([0, T])$.
- **Information:** call options $\mathcal{X} = \{(S_T - K)^+ : K \geq 0\}$.
- **Rules:** *no frictions*, dynamic in $S$, static in calls.
- Suppose we have a market model: $C(K) = \mathbb{E}(S_T - K)^+, K \geq 0$; i.e. $(S_t : t \leq T)$ is a UI martingale, $S_T \sim \mu$, $\mu(dx) = C''(x)dx$.
- Via Dubins-Schwarz $S_t = B_{\tau_t}$ is a time-changed Brownian motion. Say we have $O_T = O(S)_T = O(B)_{\tau_T}$.
- We are led then to investigate the bounds

$$LB = \inf_{\tau} \mathbb{E}O(B)_\tau, \quad \text{and} \quad UB = \sup_{\tau} \mathbb{E}O(B)_\tau,$$

for all stopping times $\tau$: $B_\tau \sim \mu$ and $(B_{t\wedge \tau})$ a UI martingale, i.e. for all solutions to the Skorokhod Embedding problem.
- Analysing the hedging strategy in the extreme model one guesses the general model-independent superhedge.
- No-duality gap: the process $S_t := B_{\tau^*_t \wedge \tau / \tau - t}$ matches information and attains a perfect hedge.
Continuous time models – SEP & one marginal

- **Beliefs:** $\mathcal{F} = C([0, T])$.
- **Information:** call options $\mathcal{X} = \{(S_T - K)^+ : K \geq 0\}$.
- **Rules:** *no frictions*, dynamic in $S$, static in calls.
- Suppose we have a market model: $C(K) = \mathbb{E}(S_T - K)^+, K \geq 0$; i.e. $(S_t : t \leq T)$ is a UI martingale, $S_T \sim \mu$, $\mu(dx) = C''(x)dx$.
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- Analysing the hedging strategy in the extreme model one guesses the general model-independent superhedge.
- No-duality gap: the process $S_t := B_{\tau^*\wedge \frac{t}{T-t}}$ matches information and attains a perfect hedge.
Continuous time models – SEP & one marginal

• **Beliefs:** \( \mathcal{P} = C([0, T]) \).
• **Information:** call options \( \mathcal{X} = \{(S_T - K)^+ : K \geq 0\} \).
• **Rules:** no frictions, dynamic in \( S \), static in calls.
• Suppose we have a market model: \( C(K) = \mathbb{E}(S_T - K)^+, K \geq 0 \); i.e. \( (S_t : t \leq T) \) is a UI martingale, \( S_T \sim \mu, \mu(dx) = C''(x)dx \).
• Via Dubins-Schwarz \( S_t = B_{\tau_t} \) is a time-changed Brownian motion. Say we have \( O_T = O(S)_{\tau} = O(B)_{\tau T} \).
• We are led then to investigate the bounds

\[
LB = \inf_{\tau} \mathbb{E}O(B)_{\tau}, \quad \text{and} \quad UB = \sup_{\tau} \mathbb{E}O(B)_{\tau},
\]

for all stopping times \( \tau: B_{\tau} \sim \mu \) and \( (B_{t \wedge \tau}) \) a UI martingale, i.e. for all solutions to the Skorokhod Embedding problem.
• Analysing the hedging strategy in the extreme model one guesses the general model-independent superhedge.
• No-duality gap: the process \( S_t := B_{\tau^* \wedge \frac{t}{T-t}} \) matches information and attains a perfect hedge.
Continuous time models – SEP & one marginal

- **Beliefs:** $\mathcal{P} = C([0, T])$.
- **Information:** call options $\mathcal{X} = \{(S_T - K)^+ : K \geq 0\}$.
- **Rules:** *no frictions*, dynamic in $S$, static in calls.
- Suppose we have a market model: $C(K) = \mathbb{E}(S_T - K)^+, \ K \geq 0$; i.e. $(S_t : t \leq T)$ is a UI martingale, $S_T \sim \mu, \mu(dx) = C''(x)dx$.
- Via Dubins-Schwarz $S_t = B_{\tau_t}$ is a time-changed Brownian motion. Say we have $O_T = O(S)_T = O(B)_{\tau_T}$.
- We are led then to investigate the bounds

\[ LB = \inf_{\tau} \mathbb{E}O(B)_{\tau}, \quad \text{and} \quad UB = \sup_{\tau} \mathbb{E}O(B)_{\tau}, \]

for all stopping times $\tau$: $B_\tau \sim \mu$ and $(B_{t \wedge \tau})$ a UI martingale, i.e. for all solutions to the Skorokhod Embedding problem.

- Analysing the hedging strategy in the extreme model one guesses the general model-independent superhedge.
- No-duality gap: the process $S_t := B_{\tau^*_\wedge \tau, \tau-t}$ matches information and attains a perfect hedge.
Continuous time models – SEP & one marginal

- **Beliefs:** \( \mathcal{P} = C([0, T]) \).
- **Information:** call options \( \mathcal{X} = \{(S_T - K)^+ : K \geq 0\} \).
- **Rules:** *no frictions*, dynamic in \( S \), static in calls.
- Suppose we have a market model: \( C(K) = \mathbb{E}(S_T - K)^+ \), \( K \geq 0 \);
  i.e. \( S_t : t \leq T \) is a UI martingale, \( S_T \sim \mu \), \( \mu(dx) = C''(x)dx \).
- Via Dubins-Schwarz \( S_t = B_{\tau_t} \) is a time-changed Brownian motion.
  Say we have \( O_T = O(S)_T = O(B)_{\tau_T} \).
- We are led then to investigate the bounds

\[
LB = \inf_{\tau} \mathbb{E}O(B)_{\tau}, \quad \text{and} \quad UB = \sup_{\tau} \mathbb{E}O(B)_{\tau},
\]

for all stopping times \( \tau : B_{\tau} \sim \mu \) and \( (B_t^{\wedge \tau}) \) a UI martingale,
  i.e. for all solutions to the Skorokhod Embedding problem.
- Analysing the hedging strategy in the extreme model one guesses the general model-independent superhedge.
- **No-duality gap:** the process \( S_t := B_{\tau^* \wedge \frac{t}{1-t}} \) matches information and attains a perfect hedge.
Continuous time models – SEP & one marginal

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Suppose we have a market model: \( C_i(K) = \mathbb{E} (S_{T_i} - K)^+ , K \geq 0 \); i.e. \( (S_t : t \leq T_n) \) is a UI martingale, \( S_{T_i} \sim \mu_i, \mu_i(dx) = C_i''(x)dx \).

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Henry-Labordère, O., Spoida and Touzi (13): stochastic control methods used to derive \( UB \) for lookback options.

O. and Spoida (13): the optimal \( n \)-marginal Azéma–Yor (like) embedding obtained.
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So far we only had *market information = prices of liquid options*. What about time-series of past data? Way may want to use it to shrink $\mathcal{P}$ and narrow down the super/sub-hedging bounds derived so far (Mykland (03,05)).

Consider $\mathcal{P}_\Xi$ the set of continuous functions $(S_t)$ which admit quadratic variation and with $[\ln S]_T \leq \Xi$. We can solve the problem of robust pricing and hedging of a European option with a convex payoff $G(S_T)$ given prices of $n$ co-maturing put options.
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$$\sup \{ \mathbb{E} G(S_T) : [\ln S]_T \leq \Xi, \mathbb{E}(K_i - S_T)^+ = P_i \}$$

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which is solved by Root’s stopping time (cf. Rost ’76).

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We consider now bounds on (future) implied volatility.

Essentially we propose a robust version of market models, cf. Schonbucher (99), Bergomi (05), Schweizer and Wissel (08), Jacod and Protter (10), Carmona and Nadtochiy (09,11).

\( \mathcal{P} \) is now the space for paths of the underlying and some options written on it and incorporates

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Our methods work pathwise and hence in any model which satisfies the beliefs.

We want to interpolate between model-free and model-dependent results:

- relaxing the beliefs we recover the model-free results
- strengthening the beliefs we obtain the model-specific results.

It is not hard to see that one can prescribe pathspace $\mathcal{P}$ which implies the Black-Scholes model.

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In nutshell the idea is: combine fully robust (model-free) hedge with static hedging of Carr and Nadtochiy (2011), i.e. find a diffusion model in which the implied volatility takes its “extremal values” and work out static hedge in that model.

Recall: if a model is fixed, a perfect static hedge of an up-and-out put corresponds to finding $g$ such that, for all $0 < t < T$:

$$(K - S_T)^+ - g(S_T), \text{ s.t. } g|_{[0,B]} = 0 \text{ and } \mathbb{E}[(K - S_T)^+ - g(S_T) | S_t = B] = 0.$$ 

In particular,

$$\mathbb{E}[(K - S_T)^+ 1_{\sup_{u \leq T} S_u < B}] = \mathbb{E}[(K - S_T)^+ - g(S_T)].$$
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![Graph showing implied variance and boundary values.](image)
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- We hedge if the barrier is \textit{not} hit.
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- Likewise we can build a subhedging strategy.
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Example: Up-and-out put. Robust superhedge.

Consider an up-and-out put option with strike $K$ and barrier $B > K$. Assume prices of vanilla put options are known with strikes $(K_i)$. For $L = K_i < K$ we have, with $\bar{S}_t := \sup_{u \leq t} S_u$,

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(K - S_T)^+ 1_{\bar{S}_T < B} \leq \frac{B - K}{B - L} (L - S_T)^+ - \frac{K - L}{B - K} (S_T - B) + \frac{K - K_i}{B - K} (S_T - B) 1_{\bar{S}_T \geq B}
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We now combine this with the robust hedge of Brown, Hobson and Rogers (01). Consider $L \leq K$ and the associated $g_1$. Then

$$\frac{B - K}{B - L}(L - S_T)^+ - \frac{K - L}{B - K}(S_T - B) + \frac{K - L}{B - K}(S_T - B)1_{S_T \geq B} - \tilde{g}_1(S_T)$$

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In general, these static robust hedges are not optimal. However they are asymptotically optimal and recover the fully robust (no beliefs) in the limit:

**Theorem (subject to assumptions...)**

If $IV \to 0$ and $IV \to \infty$ then $\tilde{g}_1 \to 0$, the robust static hedge and its cost converges to the robust hedge cost.
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Robust Static hedging – examples

Up-and-out put with $K = 0.6$, $B = 1.2$, $K_1 = 0.3$ and $\overline{IV}/\overline{IV} = 3$. 

![Graph showing the payoffs and hedging strategies for an up-and-out put option with different values for $K$, $B$, and IV. The graph compares the original robust hedge with the static robust hedge, illustrating the effectiveness of the latter in managing risk.]
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Consider a classical model with constant interest rate $r$ and price process following a time homogeneous diffusion

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t$$

Static hedging corresponds to reflection principle for diffusions:

Find $g$ s.t. $g|_{[0,B]} \equiv 0$ and $\mathbb{E}\left[(K-S_\tau)^+-g(S_\tau)|S_0 = B\right] = 0, \forall \tau \geq 0$.

An up-and-out put option has payoff $(K-S_T)^+1_{\max_{t\leq T}S_t \leq B}$. This is hedged perfectly by European option with payoff

$$G(S_T) = (K-S_T)^+ - g(S_T).$$

To compute the mirror image $g$ of the put payoff consider $\psi^1, \psi^2$ the fundamental solutions of the associate Sturm-Liouville equation:

$$\frac{1}{2}\sigma^2(s)\psi_{ss}(s,z) - (z^2 + r)\psi(s,z) = 0.$$
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An up-and-out put option has payoff $(K - S_T)^+ \mathbf{1}_{\max_{t \leq T} S_t \leq B}$. This is hedged perfectly by European option with payoff

$$G(S_T) = (K - S_T)^+ - g(S_T).$$

To compute the mirror image $g$ of the put payoff consider $\psi^1, \psi^2$ the fundamental solutions of the associate Sturm-Liouville equation:

$$\frac{1}{2} \sigma^2(s) \psi_{ss}(s, z) - (z^2 + r) \psi(s, z) = 0.$$
### Static Hedging of Carr and Nadtochiy (2011)

Consider a classical model with constant interest rate $r$ and price process following a time homogeneous diffusion

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t$$

Static hedging corresponds to **reflection principle for diffusions**:

Find $g$ s.t.

$$g\big|_{[0,B]} \equiv 0 \quad \text{and} \quad \mathbb{E}\left[(K - S_\tau)^+ - g(S_\tau)\big| S_0 = B\right] = 0, \forall \tau \geq 0.$$

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$$g(s) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\psi_1(s, z)\psi_1(K, z)}{\psi_1(B, z) - \psi_2(B, z)} \frac{dz}{z}.$$ 

In general this might be hard to compute. Here we propose to consider local volatility models where

- $\sigma(s)$ is piece-wise constant. In this case $\psi_1, \psi_2$ can be computed as linear combinations of exponentials on each subinterval.
- Further, the above integral is absolutely convergent allowing for arbitrary precise numerical approximations.
- We suggest this family is rich enough and sufficient for all practical purposes
  - any diffusion coefficient can be approximated with a piece-wise constant function,
  - any arbitrage free combination of European options can be reproduced by a diffusion of this type (possibly run on an independent stochastic clock).
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Outline

Modelling in Mathematical Finance
   Classical modelling framework
   Towards a Robust modelling framework

Theory of robust valuation and hedging (duality)
   Abstract no-duality gap results
   Explicit solutions via SEP: one and $n$ marginals

Beliefs based on historic time series
   Confidence intervals for realised volatility
   Robust Market Models

A class of diffusion models and their static hedging

Robust hedging in practice
Performance of robust hedges (with F. Ulmer (11))

- Traders A & B buy a digital double barrier option, e.g. 
  \[ 1 \{ S_T \geq b, S_T \leq \bar{b} \}, \ 1 \{ \bar{S}_T \geq \bar{b}, S_T \leq b \}, \]  
  for an initial price \( p \).

- Underlying dynamics are unknown but calibrated to the initial surface. We will use BS, Heston, Bates and VGSV models.

- Trader A will use BS delta or delta/vega hedging with the ATM IV. Rebalancing is done
  - daily or every six hours
  - optimally: based on a bandwidth around his delta and vega positions (Whalley and Wilmott (97))

- Trader B will use the robust hedges. More precisely, for a price \( p_1 > p \) Trader B buys the superhedge \( \bar{H} \). His final payoff is given as
  \[ p - p_1 - 1 \{ S_T \geq b, \bar{S}_T \leq \bar{b} \} + \bar{H} \]
  which is bounded below by \( p - p_1 \) if the path is in \( \mathcal{P} = C([0, T]) \). The hitting times of barrier are observed exactly or monitored every six hours or daily.
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Initial data corresponds to 176 quotes on AUD/USD on 14 Jan 2010: from 5Δ put to 35Δ put, ATM and from 35Δ call to 5Δ call for 16 maturities. We assume trading in $S$ carries a 4bps transaction costs and in options a 100bps costs. Spot $S_0 = 0.9308$. 

Robust approach to Mathematical Finance

Ann Arbor, Sep 2013

Jan Obłój
Performance of robust hedges (with F. Ulmer (11))

We consider 24 scenarios:

- 4 positions (long/short in DNT, DT)
- each under 6 market scenarios.

For each scenario we run MC and generate hedging errors for

- no hedging
- Traders A using 8 hedges
- Traders B using 3 hedges.

For each out of 288 combinations, we report

- Mean, SD, Skew, Kurtosis
- Minimum, Maximum
- VaR, CVaR both at 99%
- EUM, EUH – expected exponential utility with $\gamma = 1, 2$
- CDF plot

We say that hedge $H_1$ outperforms $H_2$ if

- $H_1$ achieves lower VaR, CVaR and max loss than $H_2$
- and expected return ($H_1$) $\geq$ expected return ($H_2$).
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Hedge errors CDF, long position in a DNT option, barriers (0.85, 1.01), BS model:
Hedge errors CDF, short position in a DT option, barriers (0.85, 1.01), VGSV:
Hedge errors CDF, long position in a DT option, barriers (0.875, 0.985), VGSV:
Comparative performance of hedging methods

- Overwhelmingly, we find that the robust hedges outperform traditional hedging methods. Often lead to a dramatic reduction of risk while achieving similar or higher returns.
- This remains true even in the models with jumps (Bates, VGSV).
- “Robust hedging errors” are typically positively skewed: frequent small losses compared with some large gains. “Traditional hedging errors” are typically negatively skewed.
- SD of robust hedging errors typically 2–3 larger than of the traditional hedges.
- We assumed no interest rates: applies for currency pairs with similar domestic interest rates.
- Makes sense for singularly large position in a barrier option in uncertain market conditions.
- Robust hedges akin to the methods used by “old-school traders”
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Conclusions and further research

- I advocate a new robust framework for pricing and hedging. It is a pathwise approach which combines beliefs about possible paths together with market information.

- We expect to have no-duality gap between superhedge price and max of market model prices. Number of general or specific cases worked out.

- It seems natural to use time-series data to incorporate beliefs about possible future paths. In particular we explore bounds on total realised volatility.

- Finally, we also show how to incorporate beliefs on future values of Implied Volatility. The resulting hedging strategies combine usual robust arguments with model-specific static hedging arguments. As beliefs grow weaker or stronger we interpolate between the fully robust (model-free) methods and the model-specific techniques.
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Thank You!
This talk was based on

- work in progress with Sergey Nadtochiy and Zhaoxu Hou.